Experimental validation of the stability and performance analysis of Networked Control Systems with delays

Ing. E.M.P. van de Wiel

s031927
Preface

This report, Experimental validation of the stability and performance analysis of Networked Control Systems with delays, covers the masters thesis study of the author, which has been performed within the Dynamics and Control Group of the faculty of Mechanical Engineering at the Eindhoven University of Technology, under the supervision of prof. dr. Henk Nijmeijer. The coaches during the work presented in this thesis were dr. ir. Nathan van de Wouw and ir. Marieke Posthumus-Cloosterman, both members of the Dynamics and Control Group. I would like to thank them for the assignment and their support. I would also like to thank my relatives and friends for their support during my master’s thesis.

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Abstract

The focus of this thesis is on Networked Control Systems (NCSs). An NCS consists of a continuous-time plant that is coupled over a communication network to a discrete-time controller. The advantages of NCSs are that the usage is relatively cheap and that the system diagnosis and maintenance of both the plants and the computing hardware are easier due to the reduced system wiring. The disadvantages of using a communication network in a control-loop are network-induced delays, packet dropouts and multiple packet transmissions. Network-induced delays occur while exchanging data among devices that are connected to the shared communication network and can be either constant or time-varying. Packet dropout is the effect that a data packet does not arrive at its destination. Multiple packet transmissions are needed if the amount of data is larger than the maximum packet size. In combination with packet dropouts, this may result in delivery of only part of the data.

This research focuses on the analysis and experimental validation of a discrete-time, linear, time-invariant NCS model with constant and periodic delays. It is also assumed that packet dropouts and multiple packet transmissions do not occur. The experimental validation is performed on the PATO set-up, which is a second-order system (a single actuated inertia) on which known time-delays can be implemented to represent an NCS with either constant or periodic delays.

Based on an exact discretization at the sampling instants, a discrete-time model with a constant delay is derived that represents the PATO set-up in combination with either a state-feedback or an output-feedback controller. To predict the stability of the PATO set-up, an eigenvalue-based stability-analysis is performed. This results in a stability region that gives the range of stabilizing controllers for a given constant time-delay. The dependence of the size of the stability region on the type of controller used, the delay and the sampling-time is also studied.

Additionally, the influence of the type of controller used, the different controller-gains, the constant time-delay and the sampling-time on the performance, in terms of the settling-time and the bandwidth, is investigated. The performance is analyzed by studying the eigenvalues, Bode plots of the open-loop frequency-response and simulations. To design a controller for the PATO set-up, several criteria on the stability robustness, settling-time and bandwidth are set. This results in a range of feedback-gains for a given constant time-delay that meet these criteria.

To experimentally validate the stability and performance analysis, the PATO set-up is used. In the PATO set-up, we can only measure the position of the inertia, thus a velocity estimator is used to estimate the velocity of the inertia. Therefore, the PATO set-up with an output-feedback controller is considered for the experimental validation. It is shown that the experimental results and the results of the model-based analysis match. This shows that the derived NCS model gives an accurate description of the PATO set-up and that it can be used to predict the stability and performance of the PATO set-up.

Besides the validation of the NCS model for constant time-delays, validation of an NCS for known periodic delays is performed. Therefore, the derived discrete-time model with a constant delay is extended to describe the output-feedback controlled PATO set-up with periodic delays with a known sequence. Using an eigenvalue-based stability analysis and simulations, it is shown that a
bounded variation of the delays can result in an unstable system, although for each constant delay within these delay bounds the system is stable. This result is validated experimentally.
Samenvatting (Dutch)

In dit verslag worden netwerk geregeld systemen bestudeerd. Een netwerk geregeld systeem bestaat uit een continue-tijd systeem dat via een datacommunicatie netwerk is gekoppeld aan een discrete-tijd regelaar. De voordelen van netwerk geregeld systemen zijn het relatief goedkope gebruik en de eenvoudigere systeemdiagnose en onderhoud van het totale systeem, inclusief de hardware, ten gevolge van de verminderde bekabeling. De nadelen van het gebruik van een datacommunicatie netwerk in een regelkring zijn tijdsvertragingen veroorzaakt door het data netwerk, het verliezen van data pakketten en het gebruik van meerdere data pakketten om data te versturen. De tijdsvertragingen ontstaan tijdens het uitwisselen van data tussen de apparaten die verbonden zijn via het gedeelde datacommunicatie netwerk en kunnen constant of tijdvariërend zijn. Het verlies van data pakketten wordt veroorzaakt doordat een data pakket niet bij zijn bestemming aankomt. Het gebruik van meerdere data pakketten is nodig als de hoeveelheid data die gelijktijdig verstuurd moet worden groter is dan de maximale pakketgrootte. In combinatie met pakket verlies kan dit leiden tot het arriveren van slechts een deel van de data.

Dit onderzoek richt zich op de analyse en experimentele validatie van een discrete-tijd, lineaire, tijdsinvariant model met constante en periodieke tijdsvertragingen. Er wordt aangenomen dat het verlies van data pakketten en het gebruik van meerdere data pakketten om data te versturen niet voorkomen. De experimentele validatie wordt uitgevoerd op de PATO opstelling. Dit is een tweede orde systeem (een roterende massa) waarop bekende vertragingen geïmplementeerd kunnen worden, zodanig dat een netwerk geregeld systeem met constante of periodieke vertragingen nagebootst wordt.

Gebaseerd op een exacte discretizatie op de bemonster momenten wordt een discrete-tijd model met constante tijdsvertragingen bepaald. Dit model is een representatie van de PATO opstelling in combinatie met een toestandsterugkoppeling of een uitgangsterugkoppeling. Om de stabiliteit van de PATO opstelling te voorspellen, wordt een stabiliteitsanalyse gedaan die gebaseerd is op een eigenwaarde analyse. Dit resulteert in een stabilitatsgebied dat het gebied van stabiliserende regelaars geeft voor gegeven constante tijdsvertragingen. De afhankelijkheid van de grootte van het stabilitatsgebied op het toegepaste type regelaar, de vertraging en de bemonstertijd is ook bestudeerd.

Hiernaast, is de invloed van het toegepaste type regelaar, de verschillende regelaar-parameters, de constante vertraging en de bemonstertijd op de prestatie, in termen van ‘settling’-tijd en de bandbreedte, is onderzocht. Om een regelaar te ontwerpen voor de PATO opstelling, zijn verschillende eisen aan de robuustheid van de stabiliteit, de settling-tijd en de bandbreedte gesteld. Dit leidt tot een aantal regelaars voor een gegeven constante vertraging dat aan deze eisen voldoet.

Om de stabiliteit en de prestatie analyse experimenteel te valideren, wordt de eerder genoemde PATO opstelling gebruikt. Op de PATO opstelling kan alleen de positie van de massa gemeten worden, dus is er een schatter gebruikt voor de snelheid. Daarom worden alleen de analytische resultaten van de PATO opstelling met een uitgangsterugkoppeling experimenteel gevalideerd. Hierbij wordt aangetoond dat de gemeten resultaten overeenkomen met de analytische resultaten. Hierdoor is aangetoond dat het berekende model een accurate beschrijving geeft van de PATO
opstelling en dat het gebruikt kan worden om de stabiliteit en prestatie van de PATO opstelling te voorspellen.

Naast de validatie van het model voor netwerk geregelde systemen met een constante tijdsvertraging, wordt het model voor bekende periodieke tijdsvertragingen gevalideerd. Daarom wordt eerst het berekende discrete-tijd model met een constante tijdsvertraging uitgebreid om zo de uitgangsterugkoppeling geregelde PATO opstelling met een bekende periodieke tijdsvertraging te beschrijven. Met behulp van een stabiliteitsanalyse gebaseerd op de eigenwaarden en simulaties is aangetoond dat een periodieke variatie van de tijdsvertraging kan resulteren in een instabiel systeem, hoewel het systeem voor elke constante tijdsvertraging die binnen de minimale en maximale waarde van de periodieke tijdsvertraging valt stabiel is. Dit resultaat is tevens experimenteel gevalideerd.
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Chapter 1

Introduction

In industry, the use of Networked Control Systems (NCSs) for remote control purposes and factory automation has gained an increasing interest [11]. An NCS is a control system in which the data transfer is done over a communication network [14]. In general, an NCS consists of a continuous-time plant and a discrete-time controller. A typical set-up is depicted in Figure 1.1. This figure illustrates several controlled systems, where the information, such as measurement data and control-input data, is exchanged over a communication network among system components, such as sensors, controllers and actuators. Examples of applications of NCSs are mobile sensor networks, automated highway systems, unmanned aerial vehicles and remote surgery [11]. The advantage of an NCS is that, due to the widespread use of data networks and the availability of a lot of cheap commercial off-the-shelf hardware, the usage is relatively cheap [11]. Another advantage is that the system diagnosis and maintenance of both the plants and the computing hardware are easier, due to the reduced system wiring [14]. The disadvantages of an NCS are network-induced delays, packet dropouts and multiple packet transmissions [14], which complicate the analysis of an NCS. Network-induced delays, consisting of sensor-to-controller and controller-to-actuator delays, occur while exchanging data among devices that are connected to the shared communication network. The delays can be either constant or time-varying. Packet dropout is the effect that a data packet does not arrive at its destination. Multiple packet transmissions are needed if the amount of data is larger than the maximum packet size. In combination with packet dropouts, this may result in delivery of only part of the data.

In this research the occurrence of packet dropouts and multiple control inputs will be neglected. Moreover, the delays are assumed to be constant or time-varying according to a known sequence. Therefore, the literature on NCSs with constant or periodic time-delays will be discussed briefly. The focus is on the modeling and analysis issues with respect to stability and performance, where a

![Figure 1.1: A typical NCS set-up.](image)
Continuous-time plant and a discrete-time controller are considered. One of the first contributions on NCSs, for constant time-delays, is given in [5]. Here, a discrete-time representation of an NCS is derived that contains a time-driven sensor, a time-driven controller and an event-driven actuator. ‘Time-driven’ means that only at sampling instants information is sent and ‘event-driven’ means that a process starts immediately when an input is received. In [5], a finite-dimensional discrete-time model with constant delays is obtained, which assumes that the sampling interval of both the sensor and controller is constant and equal. However, a time-skew between the sampling instants of the sensor and controller is allowed. In [5], a stability region is presented that shows the stability of the delayed NCS as a function of the feedback gain and the time skew. Besides the results for constant delays, stability results for known periodic sequences are presented.

In [9], the linear quadratic Gaussian (LQG) optimal control problem is solved. Here, the assumptions are slightly different, compared to [5], because in [9] an event-driven controller is used. Based on these assumptions, in [9], it is concluded that, if the controller and actuator are event-driven, the computational delay can be added to the total network induced delay.

A second NCS model is described in [14] that is based on the same assumptions as in [9]. In [14], based on the stability analysis, a stability region, i.e. all stabilizing controller-gains for constant delays, is presented. This region can be used to predict the range of stabilizing feedback gains for a given constant delay. Note that the analysis is performed for a one-dimensional system with a constant sampling-time. Based on this stability region it is concluded that there are stabilizing controller-gains for a delayed system, even if the system with no delay is unstable. Based on this model, in [1], a remarkable effect of periodicity in the delays is found. It is shown that the time-variation of the delay may lead to instability even if the system is stable for each fixed delay within the upper- and lowerbound of the periodic sequence.

Previous research has concentrated on the model-based analysis of NCSs, but no experimental validation is known. This research will focus on the analysis and experimental validation of a discrete-time, linear, time-invariant NCS model as used in [9], [14]. The experimental validation is performed on the PATO set-up, which is a motor set-up used in the DCT lab of the TU Eindhoven. The advantage of the PATO set-up is that it has no network that induces unknown time-varying delays. By implementing delays between the controller and the plant, an experimental set-up with known delays is obtained that can be used to represent an NCS with constant or periodic time-delays.

1.1 Problem statement

It is well known that time-delays can degrade the performance of a control system and can even result in instability [3]. The purpose of this research is to validate the analytical stability and performance results of an NCS model with constant time-delays. First, the stability region, i.e. the range of stabilizing controller-gains for constant delays, is experimentally validated. Second, the performance, in terms of the settling-time and the bandwidth, as predicted by the NCS model, is validated experimentally. Finally, it is experimentally validated that the time-variation of the delay may lead to instability even if the system is stable for each fixed delay within the upper- and lowerbound of the periodic sequence, as was observed for an NCS model in [1].

In order to achieve these research goals, the following approach is followed. A mathematical model is derived to analyze the influence of the system parameters, i.e. the delay, the controller-gains and the sampling-time, on the stability and performance of an NCS for constant time-delays. The validation of these results is done by performing experiments on the PATO set-up. The mathematical model is extended to analyze the effects of a time-variation in the delay on the stability of an NCS and, again, these results are validated by performing experiments on the PATO set-up.
1.2 Outline of the report

In Chapter 2 the basic discrete-time NCS model is explained. Additionally, a frequency-domain model is derived. The stability analysis and the analysis of the stability robustness of the PATO set-up are described in Chapter 3 for constant time-delays and two types of controllers, i.e. a state-feedback and an output-feedback controller. The performance analysis of the NCS model in terms of the settling-time and the bandwidth are described in Chapter 4 for both the state-feedback and the output-feedback case. Chapter 5 introduces the PATO set-up and discusses the experimental implementation of the delays. Chapter 6 gives the experimental validation of the stability and performance analysis results obtained in Chapter 3 and Chapter 4. In Chapter 7 both analytical and experimental results are presented that show that periodic time-delays may lead to instability even if the system is stable for each fixed delay within the upper- and lowerbound of the periodic sequence. Conclusions and recommendations are given in Chapter 8.
Chapter 2

Modeling of NCSs with constant time-delays

In this chapter, the generic discrete-time NCS model is described and applied to the PATO set-up. Section 2.1 discusses the configuration of the NCS to be modeled as well as the assumptions on which the model is based. The generic discrete-time NCS model is derived in Section 2.2. In Section 2.3, the generic NCS models are applied to the PATO set-up. Two models are distinguished, firstly, the PATO set-up with a state-feedback controller and, secondly, the same PATO set-up with an output-feedback controller.

2.1 The modeled NCS configuration

Consider the schematic representation of an NCS as given in Figure 2.1. This figure shows that a continuous-time plant $P$ is controlled by a discrete-time controller $K$ that is connected to the plant via a communication network. The discrete-time controller computes the discrete-time control-input $u_k$ from the discrete-time error $e_k$. This controller is event-driven, which means that it starts processing immediately when an input $e_k$ is received. The discrete-time control signal $u_k$ is sent to the actuator via the communication network, which induces a controller-to-actuator delay $\tau_{ca}$. The discrete-time control-input $u_k$ is converted to the continuous-time control-input $u(t)$ via a zero-order-hold (Z.O.H.) and the plant $P$ is actuated. The continuous-time output $y(t)$ of the plant is measured with an encoder, which results in the discrete-time output $y_k$. The encoder is time-driven, which means that only at sampling instants $k$ information from the plant is sent to the controller over the network. This induces a sensor-to-controller delay $\tau_{sc}$. The discrete-time error $e_k$ is given by $e_k = r_k - y_k$, with $r_k$ the discrete-time reference signal. Note that in the remainder of this chapter it is assumed that $r_k = 0$.

\[ e_k = r_k - y_k \]

![Figure 2.1: The schematic representation of the modeled NCS.](image-url)
To derive the NCS model, the following assumptions are made:

1. **the total network induced delay** \( \tau \) **is given by** \( \tau = \tau_{sc} + \tau_{ca} \), which is allowed because the modeled NCS consists of a time-driven sensor, an event-driven controller, an event-driven actuator and because the digital control algorithm is time-invariant and static \cite{9}, \cite{14},

2. **the total network induced delay** \( \tau \) **is constant**.

3. **there is no computational delay in the controller**. Note that if computational delay occurs, this is added to the total network induced delay. This is possible since the controller and actuator are event-driven \cite{9}, \cite{14},

4. **all data arrives in one packet, thus multiple packets do not occur**,

5. **packet dropout does not occur**.

### 2.2 The generic discrete-time NCS model

Recall that in Figure 2.1, the plant \( P \) is a continuous-time, linear, time-invariant system. Its dynamics are given by:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t), \\
u(t) &= u_k, & \text{for } t \in [kh + \tau, kh + h + \tau),
\end{align*}
\]  

with \( A \in \mathbb{R}^{n \times n} \) the state matrix, \( B \in \mathbb{R}^{n \times m} \) the input matrix, \( x(t) \in \mathbb{R}^{n \times 1} \) the continuous-time states of the plant, \( u(t) \in \mathbb{R}^{m \times 1} \) the continuous-time control-input, \( y(t) \in \mathbb{R}^{p \times 1} \) the continuous-time output, \( h \) the sampling-time, \( \tau \) the constant time-delay for which holds that \( 0 \leq \tau < h \) and \( u_k \) the discrete-time equivalent of the continuous-time control-input \( u(t) \) at sampling instant \( kh \) for which holds \( u_k := u(kh) \). Using a standard result from e.g. \cite{3}, \cite{12}, the discrete-time model of system (2.1) is given by:

\[
\begin{align*}
x_{k+1} &= \Phi x_k + \Gamma_0 u_k + \Gamma_1 u_{k-1}, \\
y_k &= Cx_k,
\end{align*}
\]  

with \( \Phi = e^{Ah} \), \( \Gamma_0 = \int_0^{h-\tau} e^{As} dsB \), \( \Gamma_1 = \int_{h-\tau}^h e^{As} dsB \), \( x_k \) the discrete-time equivalent of the continuous-time state \( x(t) \) for which holds that \( x_k := x(kh) \) and \( y_k \) the discrete-time equivalent of the continuous-time output \( y(t) \) for which holds that \( y_k := y(kh) \). The discrete-time model (2.2) is obtained from an exact discretization of (2.1) on the sampling instants. Note that with ‘small delays’ we indicate time-delays smaller than the sampling-time \( h \) and with ‘large delays’ we indicate time-delays larger than the sampling-time \( h \).

If the delay is larger than or equal to the sampling-time \( h \), the generic model (2.2) has to be modified, because \( u_k \) is not available in the sampling interval \([kh, kh + h)\) \cite{12}. First, let us define:

\[
\tau^* = \tau - (d - 1)h,
\]  

with \( d = \lceil \frac{\tau}{h} \rceil \), which is the smallest integer larger than or equal to \( \frac{\tau}{h} \), and \( 0 \leq \tau^* < h \). Then, the discrete-time NCS model is given by:

\[
\begin{align*}
x_{k+1} &= \Phi x_k + \Gamma_0^* u_{k-d+1} + \Gamma_1^* u_{k-d}, \\
y_k &= Cx_k,
\end{align*}
\]  

\footnote{Note that in Chapter 7 the periodic delay case will be studied, where the corresponding changes of the models will also be discussed.}
Chapter 2: Modeling of NCSs with constant time-delays

with Φ defined in (2.2), \( \Gamma_0^* = \int_{h-t^*}^{h} e^{As}dB \) and \( \Gamma_1^* = \int_{h-t^*}^{h} e^{As}dB \), where \( \tau^* \) and \( d \) are defined in (2.3).

The control law of the linear, static, discrete-time controller can be written as:

\[
u_k = Ke_k,
\]

(2.5)

with \( K \in \mathbb{R}^{m \times n} \) the controller-gain and \( e_k \) the discrete-time error given by \( e_k = r_k - y_k \), with \( r_k \) the discrete-time reference signal and \( y_k \) the discrete-time output. Recall that \( r_k = 0 \) and that \( y_k = Cx_k \). Then, (2.5) results in:

\[
u_k = -KCx_k.
\]

(2.6)

Implementing (2.6) in (2.2) gives the closed-loop, discrete-time NCS model for small delays:

\[
\begin{bmatrix}
x_{k+1} \\
x_k
\end{bmatrix} = \Psi
\begin{bmatrix}
x_k \\
x_{k-1}
\end{bmatrix},
\]

(2.7)

with

\[
\Psi = \begin{bmatrix}
\Phi - \Gamma_0^* KC & -\Gamma_1^* KC \\
I_{n \times n} & 0_{n \times n}
\end{bmatrix},
\]

(2.8)

0_{i \times j} \in \mathbb{R}^{i \times j} \) a matrix with zeros and \( I_{i} \in \mathbb{R}^{i \times i} \) the identity matrix. If the delay is larger than the sampling-time, the closed-loop, discrete-time NCS model (2.7) has to be modified, similar to (2.4). This gives:

\[
\bar{x}_{k+1} = \Psi^* \bar{x}_k,
\]

(2.9)

with \( \Psi^* = \begin{bmatrix}
\Phi & 0_{n \times n(d-2)} & -\Gamma_0^* KC & -\Gamma_1^* KC \\
I_{nd \times nd} & 0_{n \times n}
\end{bmatrix}, \bar{x}_k = \begin{bmatrix}
x_k^T \\
x_{k-1}^T \\
\vdots \\
x_{k-d}^T
\end{bmatrix}^T, \Phi \) defined in (2.2) and with \( \Gamma_0^* \) and \( \Gamma_1^* \) defined in (2.3). Note that if \( d = 1 \), the model of (2.7) is used and that if \( d = 2 \) the matrix \( 0_{n \times n(d-2)} \) is empty. Notice that if \( \tau > 0 \) then \( n(d+1) \) state variables are used to describe the closed-loop, discrete-time model.

The models of (2.7) and (2.9) are applicable for time-domain analysis. Next, these models are used to obtain a frequency-domain model that is e.g. applicable to determine the bandwidth and the sensitivity of the controlled system. Therefore, the open-loop frequency-response function of the NCS model is determined by deriving the transfer function between the error and the output. First, the discrete-time NCS model for small delays of (2.2) is defined in the z-domain as [12]:

\[
\begin{align*}
z X(z) &= \Phi X(z) + \Gamma_0 U(z) + z^{-1} \Gamma_1 U(z), \\
Y(z) &= C X(z),
\end{align*}
\]

(2.10)

with \( X(z) = X \{ x_k \} \) the z-transform of the discrete-time state \( x_k \), which is the discrete-time equivalent of the Laplace transform for continuous-time systems [12], \( U(z) = X \{ u_k \} \) the z-transform of the discrete-time control-input \( u_k \) and \( Y(z) = X \{ y_k \} \) the z-transform of the discrete-time output \( y_k \). Next, the transfer function of the plant \( P(z) \) between the frequency-domain control-input \( U(z) \) and the frequency-domain output \( Y(z) \) can be computed for time-delays smaller than the sampling-time:

\[
Y(z) = P(z) U(z),
\]

(2.11)
2.3 The discrete-time model of the PATO set-up

The derived generic NCS models can be applied to the PATO set-up, which is described in detail in Chapter 5. The PATO set-up is used for experimental time-domain as well as frequency-domain analysis and a schematic representation is depicted in Figure 2.2. This shows a rotating inertia, which is driven by a motor with input torque \( T \). The mass rotates with an angular displacement \( \varphi(t) \) and an angular velocity \( \dot{\varphi}(t) \). The corresponding equation of motion is described by:

\[
J \ddot{\varphi}(t) = T,
\]

with the total inertia of the rotating mass and motor given by \( J = 1.30 \cdot 10^{-5} \) kgm\(^2\). The input torque of the motor is given by \( T = c_v c_t u(t) \), with the motor constants \( c_v = 0.537 \) A/V, \( c_t = 0.0525 \) Nm/A and \( u(t) \) the continuous-time control-input. The state variables are given by \( x(t) = [ \varphi(t) \ \dot{\varphi}(t) ]^T \). The dynamics of the PATO set-up are given by (2.1), with \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \),

\[
Y(z) = P^*(z)U(z),
\]

with \( P^*(z) = z^{1-d}C(zI - \Phi)^{-1}(\Gamma_0^* + z^{-1}\Gamma_1^*) \), \( \Phi \) defined in (2.2) and \( \Gamma_0^* \) and \( \Gamma_1^* \) defined in (2.4). Note that for small delays, i.e. \( d = 1 \), the model of (2.11) is obtained. This means that the model of (2.12) can also be used to describe the small delay case in the frequency-domain. Next, the controller of (2.5) is expressed in the \( z \)-domain:

\[
U(z) = KE(z),
\]

with \( E(z) = Z\{e_k\} \) the \( z \)-transform of the discrete-time error \( e_k \). The open-loop frequency-response function of the NCS model for small and large delays is determined by implementing (2.13) in (2.12). This results in:

\[
Y(z) = L(z)E(z),
\]

with \( L(z) = P^*(z)K \).

Figure 2.2: The schematic representation of the PATO set-up.
Chapter 2: Modeling of NCSs with constant time-delays

\[ B = \begin{bmatrix} 0 & \frac{1}{\tau} \end{bmatrix}^T \text{ and } \frac{1}{\tau} = \frac{\omega}{\phi}. \] The output matrix \( C \) depends on the used type of controller. If a full state-feedback controller is chosen, then the system output \( y(t) \in \mathbb{R}^{n \times 1} \) and the output matrix \( C = I \); if another type of controller is chosen, e.g. an output-feedback controller, \( y(t) \) and \( C \) change. The discrete-time model of the PATO set-up for small delays is given by (2.2), with:

\[
x_k = \begin{bmatrix} \varphi_k \\ \omega_k \end{bmatrix}, \Phi = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}, \Gamma_0 = \frac{1}{J^2} \begin{bmatrix} \frac{1}{2} (h - \tau)^2 \\ h - \tau \end{bmatrix}, \Gamma_1 = \frac{1}{J^2} \begin{bmatrix} \tau (h - \frac{\tau^2}{2}) \end{bmatrix}, (2.16)
\]

\( \varphi_k := \varphi(kh) \) the discrete-time equivalent of the continuous-time, angular displacement \( \varphi(t) \) and \( \omega_k := \omega(kh) \) the discrete-time equivalent of the continuous-time, angular velocity \( \dot{\omega}(t) \). The discrete-time model of the PATO set-up for large delays is given by (2.4), with \( \Phi \) defined in (2.16) and with:

\[
\Gamma_0 = \frac{1}{J^2} \begin{bmatrix} \frac{1}{2} (h - \tau^*)^2 \end{bmatrix}, \Gamma_1 = \frac{1}{J^2} \begin{bmatrix} \tau^* (h - \frac{\tau^*}{2}) \end{bmatrix}, (2.17)
\]

where \( \tau^* \) and \( d \) are defined in (2.3).

To study the PATO set-up in the frequency-domain, the transfer function between the control-input \( U(z) \) and the system output \( Y(z) \) for small and large delays is described by (2.12) with \( \Phi \) defined in (2.16), \( \Gamma_0^* \) and \( \Gamma_1^* \) defined in (2.17) and where \( C \) depends on the used type of controller.

Based on these PATO models, in which the controller has not been specified yet, two cases for the controller will be considered. First, an NCS with a state-feedback controller and, second, with an output-feedback controller are described.

### 2.3.1 State-feedback control

For the state-feedback case, the output matrix satisfies \( C = I \). The continuous-time output satisfies \( y(t) = \begin{bmatrix} y_x(t) & y_v(t) \end{bmatrix}^T \), with \( y_x(t) \) the continuous-time angular position output and \( y_v(t) \) the continuous-time angular velocity output. The discrete-time output satisfies \( y_k = \begin{bmatrix} y_{x,k} & y_{v,k} \end{bmatrix}^T \), with \( y_{x,k} := y_x(kh) \) the discrete-time equivalent of the continuous-time angular position output \( y_x(t) \) and \( y_{v,k} := y_v(kh) \) the discrete-time equivalent of the continuous-time angular velocity output \( y_v(t) \). The discrete-time error satisfies \( e_k = \begin{bmatrix} e_{x,k} & e_{v,k} \end{bmatrix}^T \), with \( e_{x,k} \) the discrete-time angular position error and \( e_{v,k} \) the discrete-time angular velocity error. The discrete-time control law is given by (2.9), where \( x_k \) is defined in (2.10), \( C = I \) and

\[
K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}, (2.18)
\]

with \( k_1 \) and \( k_2 \) the controller-gains. Notice that (2.6), (2.18) (with \( C = I \)) represents a linear, static, discrete-time PD-controller. The closed-loop, discrete-time PATO model for small delays is given by (2.7), (2.8) with \( \Phi \), \( \Gamma_0 \) and \( \Gamma_1 \) defined in (2.16), \( C = I \) and \( K \) defined in (2.18).

If the delay is larger than the sampling-time, the closed-loop, discrete-time PATO model is given by (2.9), with \( \Phi \) defined in (2.16), \( \Gamma_0^* \) and \( \Gamma_1^* \) defined in (2.17), \( C = I \) and \( K \) defined in (2.18).

To perform frequency-domain analysis, the open-loop frequency-response function of the PATO model is described by the transfer function between the frequency-domain error \( E(z) \) and the frequency-domain output \( Y(z) \). The transfer function is given by (2.14), with \( \Phi \) defined in (2.16), \( \Gamma_0^* \) and \( \Gamma_1^* \) defined in (2.17), \( C = I \) and \( K \) defined in (2.18). This gives:

\[
L(z) = \frac{z^{1-d}}{J^2} \begin{bmatrix} k_1 \eta(z) & k_2 \eta(z) \\ k_1 \theta(z) & k_2 \theta(z) \end{bmatrix}, (2.19)
\]
with
\[ \eta(z) = \frac{1}{2} (h - \tau^*) z^2 + \frac{(h \tau^* - (\tau^*)^2 + \frac{1}{2} h^2)}{z^2 - 2z^2 + z} z + \frac{1}{2} (\tau^*)^2, \]
\[ \theta(z) = \frac{(h - \tau^*) z + \tau^*}{z^2 - z}, \]  
(2.20)

\( \tau^* \) and \( d \) defined in (2.23), \( Y(z) = \begin{bmatrix} Y_\eta(z) & Y_\theta(z) \end{bmatrix}^T \), \( Y_\eta(z) = \mathcal{Z}\{y_{\eta,k}\} \) the \( z \)-transform of the discrete-time angular position output \( y_{\eta,k} \), \( Y_\theta(z) = \mathcal{Z}\{y_{\theta,k}\} \) the \( z \)-transform of the continuous-time angular position error \( e_\theta(t) \). In this way a \( 2 \times 2 \) MIMO model is obtained, which is expected because there are two inputs, \( E_\eta(z) \) and \( E_\theta(z) \), and two outputs, \( Y_\eta(z) \) and \( Y_\theta(z) \). Unfortunately, frequency-domain analysis becomes more involved due to the MIMO nature of system (2.19). Due to interactions between the input and the output, a change in one of the inputs will generally affect all of the outputs. Therefore, it is difficult to perform frequency-domain analysis of MIMO systems [10].

In continuous-time the angular velocity error \( e_\varphi(t) \) is obviously defined as the time-derivative of the continuous-time, angular position error \( e_\theta(t) \) \( (e_\varphi(t) = \frac{de_\theta(t)}{dt}) \). In discrete-time the time-derivative is derived with substitution rules [2]. Here, the forward Euler rule, given in discrete-time by \( e_{\varphi,k} = \zeta_{\varphi,k+1} - e_{\varphi,k} \), is chosen to derive the angular velocity error in terms of the angular position error and vice versa. The forward Euler rule is given in the \( z \)-domain by \( E_\varphi(z) = \frac{z - 1}{h} E_\theta(z) \). When we apply such a substitution to (2.19) the following open-loop frequency-response function of the state-feedback controlled PATO model is obtained:

\[ Y(z) = \tilde{L}(z) E(z), \quad (2.21) \]

with \( \tilde{L}(z) = \begin{bmatrix} \tilde{L}_1(z) & 0 \\ 0 & \tilde{L}_2(z) \end{bmatrix} \), \( \tilde{L}_1(z) = \frac{z^{1-a}}{\tau^*} \left( k_1 + k_2 \frac{z - 1}{h} \right) \eta(z), \)
\( \tilde{L}_2(z) = \frac{z^{1-d}}{\tau^*} \left( k_1 \frac{h}{z - 1} + k_2 \right) \theta(z) \)

and with \( \eta(z) \) and \( \theta(z) \) defined in (2.19). This \( 2 \times 2 \) MIMO model can be treated as two SISO models, because there are no off-diagonal terms [11]. Because (2.21) is not an exact model of the PATO set-up (due to the error induced by the forward Euler rule), differences may appear in the results for the time-domain and the frequency-domain analysis. These differences are discussed in detail in Paragraph 3.2.1 and in Paragraph 4.2.1.

### 2.3.2 Output-feedback control

In practice, often, only the position is measured and output-feedback control is used to stabilize the system, so the output matrix satisfies \( C = \begin{bmatrix} 1 & 0 \end{bmatrix} \). Now, the continuous-time output is given by \( y(t) = y_\eta(t), \) the discrete-time output by \( y_k = y_{\eta,k} \) and the discrete-time error by \( e_k = e_{\varphi,k} \).

In the output-feedback controller the velocity is estimated with the backward Euler method, i.e. \( e_{\varphi,k} = \frac{e_{\varphi,k} - e_{\varphi,k-1}}{h} \). The PD-type control law is then given by:

\[ u_k = k_1 e_{\varphi,k} + k_2 \frac{e_{\varphi,k} - e_{\varphi,k-1}}{h} \]

(2.22)

with \( k_1 \) and \( k_2 \) the controller-gains. Using \( e_k = r_k - y_k, r_k = 0, y_k = C x_k \) and \( x_k \) defined in (2.10), this results in:

\[ u_k = - \left( k_1 \varphi_k + k_2 \frac{\varphi_k - \varphi_k-1}{h} \right), \]

(2.23)

with \( \varphi_k \) the discrete-time angular displacement of the inertia. Consequently, the closed-loop, discrete-time model for small delays is obtained by implementing (2.22) in (2.2), with \( C = \begin{bmatrix} 1 & 0 \end{bmatrix} \)
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and $\Phi$, $\Gamma_0$ and $\Gamma_1$ defined in (2.16). This results in:

$$\xi_{k+1} = \Psi \xi_k,$$

(2.24)

with

$$\Psi = \begin{bmatrix} \Phi & 0_{2 \times 1} & 0_{2 \times 1} \\ 1 & 0 & 0_{1 \times 2} \\ 0_{1 \times 2} & 1 & 0 \end{bmatrix} + \begin{bmatrix} \alpha & 0_{2 \times 1} & \beta & \gamma \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

(2.25)

$$\alpha = -\frac{1}{J^*} \left[ \frac{1}{2} \left( k_1 + \frac{k_2}{h} \right) (h - \tau)^2 \right], \quad \beta = \frac{1}{J^*} \left[ \frac{k_2}{h} (h - \tau)^2 - \tau (k_1 + \frac{k_2}{h}) (h - \frac{\tau}{2}) \right],$$

$$\gamma = \frac{1}{J^*} \left[ \frac{k_2}{h} (h - \frac{\tau}{2}) \right],$$

$$\xi_k = \begin{bmatrix} x_k & \varphi_{k-1} & \varphi_{k-2} \end{bmatrix}^T$$

and $x_k$ defined in (2.16). If the time-delay is larger than the sampling-time, the discrete-time model (2.24) has to be modified, similar to (2.3). This gives:

$$\xi_{k+1} = \Psi^* \xi_k,$$

(2.26)

with:

$$\Psi^* = \begin{bmatrix} \Phi & 0_{2 \times d-2} & \alpha^* & \beta^* & \gamma^* \\ 1 & 0_{1 \times d} & 0 & 0_{d \times d} & I_{d \times d} \end{bmatrix},$$

(2.27)

$$\xi_k = \begin{bmatrix} x_k^T & \varphi_{k-1} & \ldots & \varphi_{k-d} & \varphi_{k-d-1} \end{bmatrix}^T, \quad 0_{i \times j} \in \mathbb{R}^{i \times j} \text{ a matrix with zeros, } I_{i \times i} \in \mathbb{R}^{i \times i} \text{ an identity matrix and } \alpha^*, \beta^* \text{ and } \gamma^* \text{ defined equal to } \alpha, \beta \text{ and } \gamma \text{ in (2.25)} \text{ but with } \tau \text{ replaced by } \tau^*. \text{ Different to (2.27), (2.28), } n + 1 + d \text{ state variables are used to describe the closed-loop, discrete-time PATO model. This difference is caused by using the angular displacement and the backward Euler approach in the control law of the output-feedback controller.}

The models of (2.24) and (2.26) are applicable for time-domain analysis. To perform frequency-domain analysis, the open-loop frequency-response function of the PATO model is determined by deriving the transfer function between the frequency-domain error $E(z)$ and the frequency-domain output $Y(z)$. Since only the position error is used to derive the control-input, the output-feedback controller of (2.22) is described in the $z$-domain by the transfer function between the frequency-domain, angular position error $E_x(z)$ and the frequency-domain control-input $U(z)$ by:

$$U(z) = K(z)E_x(z),$$

(2.28)

with $K(z) = k_1 + k_2 \frac{z}{h}$ and $E_x(z)$ the frequency-domain angular position error. The open-loop frequency-response function of the output-feedback controlled PATO model for small and large delays is obtained by combining (2.25) and (2.12) with $C = [1 \ 0]$, $d$ and $\tau^*$ defined in (2.3), $\Phi$ defined in (2.16) and with $\Gamma_0$ and $\Gamma_1$ defined in (2.17). This results in:

$$Y_x(z) = L(z)E_x(z),$$

(2.29)

with $L(z) = P^*(z)K(z)$, $P^*(z) = \frac{z^{1-d}}{J^*} \left[ \frac{\frac{1}{2} (h - \tau)^2 z^2 + (h \tau^* - \tau^*)^2 + \frac{1}{2} h^2 z + \frac{1}{2} (\tau^*)^2}{z^2 - 2z + 1} \right]$ and $K(z)$ defined in (2.28). In this way a SISO model is obtained.
2.3 The discrete-time model of the PATO set-up
Chapter 3

Stability analysis for constant time-delays

Based on the models derived in Chapter 2, the stability of the PATO set-up is investigated to determine which controller settings can be used for the PATO set-up. This is described in Section 3.1. In practice, the parameters of the PATO set-up may change, which may influence the stability of the PATO set-up. Therefore, the robustness of the stability to parameter changes is studied in Section 3.2. In Section 3.3, the results on the stability and the robustness of the stability are compared to determine which controller settings can be used for the PATO set-up. In each section, the influence of a different choice of the constant time-delay \( \tau \), the sampling-time \( h \), the controller-gain \( k_1 \) (the position feedback gain) and the controller-gain \( k_2 \) (the velocity feedback gain) is investigated for the state- and the output-feedback case.

3.1 Stability analysis

The stability of the PATO set-up is investigated by studying if all eigenvalues \( \lambda_i \) of the discrete-time PATO model in closed-loop are located inside the unit-circle \([3]\). The stability is analyzed in terms of its dependency on a constant time-delay \( \tau \) and the controller-gain \( k_2 \). Additionally, the influence of a different choice of the controller-gain \( k_1 \) and the sampling-time \( h \) are investigated.

The stability is analyzed for the state-feedback case in Paragraph 3.1.1 and the output-feedback case in Paragraph 3.1.2.

3.1.1 State-feedback control

For the state-feedback case, the closed-loop, discrete-time model for the PATO set-up, given by (2.7) for small delays \( (\tau < h) \) and by (2.9) for large delays \( (\tau \geq h) \), is considered. Here, the controller \( K \) is defined in (2.18), \( \Phi, \Gamma_0 \) and \( \Gamma_1 \) are defined in (2.16), \( \Gamma_0^* \) and \( \Gamma_1^* \) are defined in (2.17) and \( C = I \), i.e. all states are measured. First, all stabilizing controller-gains \( k_2 \) are determined for different values of the constant time-delay \( \tau \), with the controller-gain \( k_1 = 17 \) V/rad and the sampling-time \( h = 2 \) ms. The obtained minimum and maximum stabilizing values of \( k_2 \) are described by the solid line in Figure 3.1 where the delay is depicted as a fraction of the sampling-time (i.e. \( \frac{\tau}{h} \)). The delay as a fraction of the sampling-time is denoted as the relative delay \( \frac{\tau}{h} \). The region enclosed by this solid line (including the line itself) gives all \( k_2 \) values that guarantee stability for the PATO set-up for a given constant time-delay \( \tau \). In the remaining part of this report, this region is denoted as the stability region. From this stability region it can be concluded
3.1 Stability analysis

Figure 3.1: The stability region in terms of controller-gain $k_2$ and constant time-delay $\tau$ for several controller-gains $k_1$ and a sampling-time $h = 2$ ms.

that above a certain value of $\tilde{\tau}$ no stabilizing controllers can be found. Figure 3.1 also shows that the maximum stabilizing value of $k_2$ for $\tilde{\tau} = 0.25$ is larger than the maximum stabilizing value of $k_2$ for $\tilde{\tau} = 0$, which is denoted as the stability peak. This means that time-delays can have a stabilizing effect, which is opposite to the case of continuous-time linear, time-invariant (LTI) systems where time-delays are always deteriorating the stability. This stability peak can be explained by studying the open-loop frequency-response function of the state-feedback controlled PATO model (2.21) for several constant time-delays. Recall that description (2.21) is an estimation of the open-loop frequency-response function (2.19), which is a multiple-input multiple-output (MIMO) system. Figure 3.2 shows the open-loop frequency-response function, which describes the transfer function $\tilde{L}(z)$ between the error and the system output. Note that the equivalent characteristics in the $z$-domain are given by the expression $z = e^{j2\pi fh}$, with $f$ the frequency and $h$ the sampling-time. Since the off-diagonal terms of (2.21) are zero, only the diagonal terms are shown. Figure 3.2(a) depicts the open-loop frequency-response function $\tilde{L}_1(z)$ of the transfer function between the angular position error $E_x(z)$ and the angular position output $Y_x(z)$ for several constant time-delays. All of them are stable, because for a magnitude $|\tilde{L}_1(z)| \geq 1$, it holds that the corresponding angle $\angle \tilde{L}_1(z) > -180^\circ$. The open-loop frequency-response function $\tilde{L}_2(z)$ of the transfer function between the angular velocity error $E_v(z)$ and the angular velocity output

Figure 3.2: The open-loop frequency-response function $\tilde{L}(z)$ for several constant time-delays $\tau$, controller-gains $k_1 = 17$ V/rad and $k_2 = 0.6$ Vs/rad and a sampling-time $h = 2$ ms.
Regions are depicted in Figure 3.1 for a sampling-time \( h \).

Details on the Jury test are given in Appendix A. According to the Jury test it should hold

\[
T \geq 0.2\text{ms}
\]

To study the influence of the controller-gain \( k \) for \( \tau \), several values are chosen. The obtained stability regions are depicted in Figure 3.4 for a sampling-time \( h = 2 \) ms. This shows that the stability region decreases if the controller-gain \( k_1 \) increases, but the influence of a different choice of \( k_1 \) on the stability region is relatively small. As determined with the Jury test for \( \tau = 0 \), the corresponding minimum stabilizing value of \( k_2 \) increases as the value of \( k_3 \) increases and the maximum stabilizing value of \( k_2 \), for \( \tau = 0 \), is not affected by the choice of \( k_1 \).

Besides the influence of controller-gain \( k_1 \), the influence of the sampling-time \( h \) on the stability is investigated. The obtained stability regions for several values of \( h \) and a controller-gain \( k_1 = 17 \text{ V/rad} \) are depicted in Figure 3.3. This shows that the stability region decreases if the sampling-time \( h \) increases. If the sampling-time \( h \) increases, then the control-input is adjusted less times per second. This results in too few changes of the control-input to keep the system stable.

Besides the stability peak, another remarkable aspect is the existence of a left stability boundary in Figure 3.1. This means that some level of velocity feedback is needed, i.e. it should hold that \( k_2 \geq k_3 \), with \( k_3 \) the minimum value of \( k_c \) for which stability is guaranteed. To understand this lower-bound, the Jury test is applied for \( \tau = 0 \) to determine the lower- and upper-bound for \( k_2 \) analytically. This is performed for \( \tau = 0 \), because analysis in terms of \( \tau \) is too complicated.

Details on the Jury test are given in Appendix A. According to the Jury test it should hold for \( k_3 \), if \( \tau = 0 \), that

\[
\frac{h_{k_1}}{2} \leq k_2 < \frac{2h_{\tau}}{1}
\]

This shows that the lower-bound depends on \( k_1 \) and \( h \) and that the upper-bound depends on \( h \) and is independent of \( k_1 \). For the PATO set-up, with sampling-time \( h = 2 \) ms, controller-gain \( k_1 = 17 \text{ V/rad} \) and inertia \( J^* = 4.61 \cdot 10^{-4} \text{ V rad/s}^2 \), this gives: \( 0.017 < k_2 < 0.461 \text{ Vs/rad} \), which corresponds with the bounds depicted in Figure 3.1.

To study the influence of the controller-gain \( k_1 \), several values are chosen. The obtained stability regions are depicted in Figure 3.1 for a sampling-time \( h = 2 \) ms. This shows that the stability region decreases if the controller-gain \( k_1 \) increases, but the influence of a different choice of \( k_1 \) on the stability region is relatively small. As determined with the Jury test for \( \tau = 0 \), the corresponding minimum stabilizing value of \( k_2 \) increases as the value of \( k_3 \) increases and the maximum stabilizing value of \( k_2 \), for \( \tau = 0 \), is not affected by the choice of \( k_1 \).

Besides the influence of controller-gain \( k_1 \), the influence of the sampling-time \( h \) on the stability is investigated. The obtained stability regions for several values of \( h \) and a controller-gain \( k_1 = 17 \text{ V/rad} \) are depicted in Figure 3.3. This shows that the stability region decreases if the sampling-time \( h \) increases. If the sampling-time \( h \) increases, then the control-input is adjusted less times per second. This results in too few changes of the control-input to keep the system stable.

\[ Y_c(z) \] is depicted in Figure 3.2(b) This shows that if the time-delay increases, the phase lag of \( L_2(z) \) also increases. Besides the increase of the phase lag, the magnitude of \( L_2(z) \) decreases.

This differs from continuous-time LTI systems with time-delay, where only the phase is affected by time-delays and not the magnitude, which leads to a destabilizing effect if the time-delay increases.

In discrete-time, see Figure 3.2(b), the decrease of the magnitude of \( L_2(z) \) for values of the relative delay \( \frac{\tau}{h} \) is large enough to result in stability for \( \frac{\tau}{h} = 0.25 \), despite the increase of the phase lag. For values of the relative delay \( \frac{\tau}{h} > 0.25 \) the increase of the phase lag of \( L_2(z) \) is larger than the decrease of the magnitude, resulting in instability for the discrete-time system with a relative delay \( \frac{\tau}{h} = 0.5 \). So it is concluded that the occurrence of the stability peak is caused by the influence of the time-delay on \( L_2(z) \).

Figure 3.3: The stability region in terms of controller-gain \( k_2 \) and constant time-delay \( \tau \) for several sampling-times \( h \) and a controller-gain \( k_1 = 17 \text{ V/rad} \).

Figure 3.4: The stability region of system (2.7) and the minimum and maximum stabilizing values of \( k_2 \) of system (2.21) in terms of controller-gain \( k_2 \) and constant time-delay \( \tau \) for a controller-gain \( k_1 = 17 \text{ V/rad} \) and a sampling-time \( h = 2 \text{ ms} \).
3.1 Stability analysis

also shows that if the value of \( h \) decreases, for higher values of the relative delay \( \frac{\tau}{h} \) stabilizing controllers can be found. The influence of \( h \) on the stability peak is such that the stability peak decreases if the value of \( h \) increases. The stability peak appears independently of \( h \) at \( \tau/h = 0 \). As was obtained with the Jury test for \( \tau = 0 \), the minimum and maximum stabilizing values of \( k_2 \) depend on \( h \).

In Chapter 2 it was stated that the open-loop frequency-response function (2.20), (2.21) is an estimation of the discrete-time PATO model (2.7), (2.8), (2.16), (2.18) for small delays or (2.3), (2.19), (2.17), (2.18) for large delays. To study the accuracy of the open-loop frequency-response function (2.20), (2.21), the minimum and maximum stabilizing values of \( k_2 \), which are given by \( k_{2\min} \) and \( k_{2\max} \), respectively, are studied for several constant delays \( \tau \) with \( k_1 = 17 \) V/rad and \( h = 2 \) ms. Together with the stability region, as determined with the eigenvalue analysis, the results are depicted in Figure 3.4. If the minimum and maximum stabilizing values of \( k_2 \) for the open-loop frequency-response function (2.20), (2.21) are compared to the stability region of system (2.7), (2.8), (2.16), (2.18) for small delays or (2.3), (2.9), (2.16), (2.17), (2.18) for large delays, it is concluded that the minimum and maximum stabilizing values of \( k_2 \) would have increased. This is caused by the forward Euler substitution rule, given in discrete-time by \( e^{v,k} = e^{x,k+1} - e^{x,k}h \), which is used in the open-loop frequency-response function (2.20), (2.21) to estimate the angular velocity error in terms of the angular position error and vice versa. If the Euler backward rule is used, then the minimum and maximum stabilizing values of \( k_2 \) would have decreased. Because the difference between the results on the open-loop frequency-response function (2.20), (2.21) and the model of (2.7), (2.8), (2.16), (2.18) for small delays or (2.3), (2.9), (2.16), (2.17), (2.18) for large delays is small, the open-loop frequency-response function (2.20), (2.21) can be used to perform frequency-domain analysis of the state-feedback controlled PATO set-up.

3.1.2 Output-feedback control

Similar to the state-feedback case, the stability of the PATO set-up with an output-feedback controller is determined based on analysis of the eigenvalues. For the output-feedback case, the closed-loop discrete-time model of the PATO set-up is defined in (2.25), with \( \tau \) replaced by \( \tau^* \) and \( \tau^* \) defined in (2.3) for large delays. First, all stabilizing values of the controller-gain \( k_2 \) in combination with a controller-gain \( k_1 = 17 \) V/rad and a sampling-time \( h = 2 \) ms, are determined for different constant delays \( \tau \). The obtained stability region is depicted in Figure 3.5. Recall that the delay is depicted as a fraction of the sampling-time. Compared to

![Figure 3.5](image-url)  
**Figure 3.5:** The stability region in terms of controller-gain \( k_2 \) and constant time-delay \( \frac{\tau}{h} \) for several controller-gains \( k_1 \) and a sampling-time \( h = 2 \) ms.

![Figure 3.6](image-url)  
**Figure 3.6:** The open-loop frequency-response function \( L(z) \) for several constant time-delays \( \frac{\tau}{h} \), controller-gains \( k_1 = 17 \) V/rad and \( k_2 = 0.6 \) Vs/rad and a sampling-time \( h = 2 \) ms.
the state-feedback case, the stability region is smaller and the stability peak is disappeared. This latter observation can be explained by studying the open-loop frequency-response function $L(z)$ of the output-feedback controlled PATO model (2.25), (2.29). Recall that this model describes the transfer function between the angular position error and the angular position output. Figure 3.6 depicts the open-loop frequency-responses functions for several constant time-delays $\tau$ in combination with the controller-gains $k_1 = 17 \text{ V/rad}$ and $k_2 = 0.6 \text{ Vs/rad}$ and the sampling-time $h = 2$ ms. Recall that the equivalent characteristics in the $z$-domain are given by $z = e^{j2\pi / h}$ [2]. This shows that, compared to Figure 3.2 (a), for an increase of the constant delay $\tau$, the increase of the phase lag of $L(z)$ leads to instability, despite the decrease of the magnitude of $L(z)$. Compared to the state-feedback case, this results in a smaller maximum stabilizing value of $k_2$, for the same constant delay $\tau$.

Similar to the state-feedback case a left stability boundary exists. Again, the Jury test is applied for a non-delayed system, i.e. $\tau = 0$, to determine the lower- and upper-bound for $k_2$ analytically. More information is found in Appendix A. According to the Jury test it should hold for $k_2$ that:

$$\frac{4J^*-1_k h^* - \sqrt{16(J^*)^2 - 24k_2 h^* 4\pi J^* + 4k_2^2 h^*}}{\pi} < k_2 < \frac{4J^*-1_k h^* + \sqrt{16(J^*)^2 - 24k_2 h^* 4\pi J^* + 4k_2^2 h^*}}{\pi}.$$  

It is concluded that both the lower- and upper-bound depend on the controller-gain $k_1$ and the sampling-time $h$. This differs from the state-feedback case, where the upper-bound is independent of the controller-gain $k_1$. For the PATO set-up, with sampling-time $h = 2$ ms, controller-gain $k_1 = 17 \text{ V/rad}$ and an inertia $J^* = 4.61 \cdot 10^{-4} \text{ V rad/s}^2$, this gives: $0.018 < k_2 < 0.426 \text{ Vs/rad}$, which corresponds to the bounds in Figure 3.5.

For different values of $k_1$, the obtained stability regions are depicted in Figure 3.3 ($h = 2$ ms). Similar to the state-feedback case, the stability region decreases as the value of $k_1$ increases. Compared to the state-feedback case, the influence of $k_1$ on the minimum stabilizing value of $k_2$ is increased. Another difference with the state-feedback case is that the maximum stabilizing value of $k_2$ for $\tau = 0$ is also dependent on $k_1$ and not only the minimum stabilizing value of $k_2$. This corresponds to the results obtained with the Jury test.

The influence of the sampling-time $h$ on the stability is also investigated. The obtained stability regions for different values of $h$ and a controller-gain $k_1 = 17 \text{ V/rad}$ are depicted in Figure 3.7. The influence of $h$ on the stability of the output-feedback controlled PATO set-up is comparable to the state-feedback case. However, for small values of $k_2$, the influence of $h$ is larger compared to the state-feedback case.

**Figure 3.7:** The stability region in terms of controller-gain $k_2$ and constant time-delay $\tau$ for several sampling-times $h$ and a controller-gain $k_1 = 17 \text{ V/rad}$.  

**Figure 3.8:** The stability region of (2.24) and the minimum and maximum stabilizing values of $k_2$ of (2.29) in terms of controller-gain $k_2$ and constant time-delay $\tau$ for a controller-gain $k_1 = 17 \text{ V/rad}$ and a sampling-time $h = 2$ ms.
In Chapter 2, it was stated that the open-loop frequency-response function (2.28), (2.29) is an exact z-transform of the discrete-time PATO model (2.24), (2.25) for small delays and (2.26), (2.27) for large delays. To show this, the minimum and maximum stabilizing values of $k_2$, given by $k_1$ and $k_2$, respectively, are studied for several constant time-delays, a controller-gain $k_1 = 17$ V/rad and a sampling-time $h = 2$ ms. These results are depicted with the stability region, as determined with the eigenvalue analysis, in Figure 3.8. This shows that the minimum and maximum stabilizing values of $k_2$ for the open-loop frequency-response function (2.28), (2.29) are equal to the stability region of system (2.24), (2.25) for small delays and (2.26), (2.27) for large delays. Therefore, it is concluded that the open-loop frequency-response function (2.28), (2.29) is an exact z-transform of the discrete-time PATO model (2.24), (2.25) for small delays and (2.26), (2.27) for large delays.

A comparison between the stability of the output-feedback case and the stability of the state-feedback case, shows that the stability peak is disappeared for the output-feedback case. For both cases, the stability region decreases if the value of $k_1$ increases and if the value of $h$ increases. For the output-feedback case, the influence of the controller-gain $k_1$ and the sampling-time $h$, for small values of $k_2$, is larger than the influence of the controller-gain $k_1$ and the sampling-time $h$ for the state-feedback case. Another difference between the state-feedback case and the output-feedback case is that, for the state-feedback case, the maximum stabilizing value of $k_2$ for $\tau = 0$ is not influenced by $k_1$, while, for the output-feedback case, the maximum stabilizing value of $k_2$ decreases if the value of $k_1$ increases. Because only the angular position of the PATO set-up is measured, the output-feedback case will be used for the experimental validation study in Chapter 6.

### 3.2 Robustness of the stability to parameter changes

In practice, the parameters used in the closed-loop, discrete-time models for the PATO set-up, given by (2.7), (2.8), (2.16), (2.18) for the state-feedback case and by (2.24), (2.25) for the output-feedback case, may differ. These differences are caused by the accuracy of the measurements, which are performed to determine the motor constants $c_v$ and $c_t$ and the total inertia $J$ and/or by measurement noise. To study the robustness of the stability to these differences, the sensitivity function $S(z)$ is studied. First, consider the schematic representation of Figure 3.9. This figure is similar to the modeled NCS as represented in Figure 2.1 but with the discrete-time output disturbance $n_k$ added to the discrete-time output $y_k$. The sensitivity function is defined as the closed-loop transfer function from the output disturbances to the system error and can be obtained from:

$$ |E(z)| = |S(z)| \cdot |N(z)|, $$

with:

$$ |S(z)| = |I + L(z)|^{-1}, $$

(3.1)

![Figure 3.9: The schematic representation of the modeled NCS.](image-url)
$E(z) = Z\{e_k\}$ the $z$-transform of the discrete-time system error $e_k$, $N(z) = Z\{n_k\}$ the $z$-transform of the output disturbance $n_k$ and where $L(z)$ depends on the used type of controller. For the state-feedback case, the estimated transfer function $\tilde{L}(z)$ in (2.20), (2.21) is used, instead of the transfer function $L(z)$ in (2.19), (2.20), to simplify the computation. For the output-feedback case, $L(z)$ is given in (2.28), (2.29).

In addition to being a factor of the system error, the sensitivity function is also the reciprocal of the distance of the Nyquist curve from the critical point $-1$. A large value for $|S(z)|$ indicates a Nyquist plot that comes close to the point of instability, which decreases the robustness of the stability. Therefore, the maximum value of the magnitude of the sensitivity function, which is denoted as the sensitivity peak, is determined to study the robustness of the stability. This is given by:

$$S_m = \max_{f \in \mathbb{R}} |S(z)|,$$

(3.2)

with the equivalent characteristics in the $z$-domain given by the expression $z = e^{j2\pi fh}$, with $f$ the frequency and $h$ the sampling-time. To ensure sufficient robustness of the stability, it is demanded that the value of the sensitivity peak $S_m \leq 2$, which is a common demand in control engineering [3]. Next, the sensitivity of the PATO set-up will be analyzed for the state-feedback and the output-feedback case.

### 3.2.1 State-feedback control

For the state-feedback case, the sensitivity function (3.1) is considered with $L(z)$ defined by $\tilde{L}(z)$ in (2.20), (2.21). To study the robustness of the stability, the sensitivity peak is determined. An example is presented in Figure 3.10, where the solid line represents the magnitude of the sensitivity function for the controller-gains $k_1 = 17$ V/rad and $k_2 = 0.1$ Vs/rad, $\tau = 0$ and the sampling-time $h = 2$ ms. The dotted line represents the sensitivity peak as described in (3.2). Recall that, due to the diagonal structure, the open-loop frequency-response function in (2.20), (2.21) is studied based on two SISO models. Therefore, the sensitivity peaks $S_{m,1}$ and $S_{m,2}$ are of interest. The robustness of the stability is determined by the largest value of $S_{m,1}$ and $S_{m,2}$, i.e. $S_m = \max\{S_{m,1}, S_{m,2}\}$, because this value is closest to the point of instability. Figure 3.10 gives that $S_{m,1} = 1.2$ and that $S_{m,2} = 1.6$, which gives that $S_m = 1.6$, which satisfies the robustness margin $S_m \leq 2$.

![Figure 3.10: The sensitivity function for controller-gains $k_1 = 17$ V/rad and $k_2 = 0.1$ Vs/rad, $\tau = 0$ and a sampling-time $h = 2$ ms.](image-url)
3.2 Robustness of the stability to parameter changes

Next, the sensitivity peak is studied in terms of $k_2$. Figure 3.11 depicts the values of $S_m$ in terms of $k_2$ for the controller-gain $k_1 = 17$ V/rad, $\tau = 0$ and the sampling-time $h = 2$ ms. This shows that the influence of $k_2$ on the sensitivity peak is such that if the value of $k_2$ increases for values of $k_2 < 0.13$ Vs/rad the sensitivity peak decreases and increases if the value of $k_2$ increases for values of $k_2 > 0.13$ Vs/rad. This means that the robustness of the stability is largest for $k_2 = 0.13$ Vs/rad. Because the sensitivity peak is the reciprocal of the distance of the Nyquist curve from the critical point $-1$, with respect to stability, the value of $S_m$ is largest near bounds of the stability region. This is indeed seen in Figure 3.11.

Several values of $k_1$ are chosen to study the influence of $k_1$ on the robustness of the stability of (2.21). The obtained values of the sensitivity peak, in terms of $k_2$, are depicted in Figure 3.12 ($\tau = 0$ and $h = 2$ ms), where the solid line corresponds to Figure 3.11. Different to the stability analysis in the previous section, it seems that the maximum stabilizing value of $k_2$ is dependent on the controller-gain $k_1$. The difference is caused by the forward Euler substitution rule, given in discrete-time by $e_{v,k} = e_{x,k+1} - \frac{e_{x,k}}{h}$, which is used in the open-loop frequency-response function (2.20), (2.21) to estimate the angular velocity error in terms of the angular position error and vice versa. Due to this substitution rule, the transfer function between the angular velocity error and

![Figure 3.11:](image1)

**Figure 3.11:** The maximum magnitude of the sensitivity $S_m$ in terms of controller-gain $k_2$ for a controller-gain $k_1 = 17$ V/rad, $\tau = 0$ and a sampling-time $h = 2$ ms.

![Figure 3.12:](image2)

**Figure 3.12:** The maximum magnitude of the sensitivity $S_m$ in terms of controller-gain $k_2$ for several controller-gains $k_1$, $\tau = 0$ and a sampling-time $h = 2$ ms.

![Figure 3.13:](image3)

**Figure 3.13:** The maximum magnitude of the sensitivity $S_m$ in terms of controller-gain $k_2$ for several sampling-times $h$ a controller-gain $k_1 = 17$ V/rad and no time-delay.
the angular velocity output is dependent on $k_1$. Figure 3.12 also shows that the sensitivity peak is larger if the value of $k_1$ is larger for small values of $k_2$, i.e. the value of $k_2$ is smaller than the value of $k_2$ for which the smallest value of the sensitivity peak is reached. For larger values of $k_2$, the sensitivity peak decreases if the value of $k_1$ increases. Note that the value of $k_2$, for which the smallest value of the sensitivity peak appears, increases if the value of $k_1$ increases.

The influence of $h$ on the sensitivity peak of (2.21) is also studied in terms of $k_2$. The results are depicted in Figure 3.13 for $\tau = 0$ and a controller-gain $k_1 = 17$ V/rad. Again, the solid line corresponds to Figure 3.11. Figure 3.13 shows that if the sampling-time increases, the sensitivity peak also increases. The value of $k_2$, for which the smallest value of the sensitivity peak appears, also increases if the value of $h$ increases.

Besides the influence of $k_1$ and $h$, the influence of the constant time-delay $\tau$ on the robustness of the stability is studied. Therefore, the obtained sensitivity peak is depicted in Figure 3.14 in terms of $k_2$ for several values of $\tau$, a controller-gain $k_1 = 17$ V/rad and a sampling-time $h = 2$ ms. The largest values of $S_m$ correspond to the stability bounds obtained in Paragraph 3.1.1 and

Figure 3.15: The area that guarantees a sufficient level of stability robustness ($S_m \leq 2$) in terms of controller-gain $k_2$ and constant delay $\tau$ for a controller-gain $k_1 = 17$ V/rad and a sampling-time $h = 2$ ms.
the solid line in Figure 3.14(a) corresponds to Figure 3.11. Figure 3.14 shows that the sensitivity peak increases if the time-delay increases, except for values of \(k_2\) near the corresponding stability bounds.

Next, the demanded level of stability robustness for model uncertainties is taken into account. The gray area in Figure 3.15 shows all values of \(k_2\) that guarantee stability but where the demanded level of stability robustness is not met, i.e. \(S_m > 2\), in terms of a constant time-delay \(\tau\), depicted as a fraction of the sampling-time, with the controller-gain \(k_1 = 17 \text{ V/rad}\) and the sampling-time \(h = 2 \text{ ms}\). Here, the solid line corresponds to the minimum and maximum stabilizing values of \(k_2\), see Paragraph 3.1.1. From Figure 3.15 it is concluded that the region which ensures sufficient robustness is much smaller than the stability region and does not allow relative delays \(\frac{\tau}{h} > 0.49\), at least for this value of \(k_1\) and \(h\). The shape of the stability peak is returned in the region where the demanded level of stability robustness is met. Therefore, it is concluded that the stability peak improves the robustness of the stability.

### 3.2.2 Output-feedback control

For the output-feedback case, the sensitivity function (3.1), with \(L(z)\) defined in (2.29), is considered. Recall that (2.29) is a SISO model and that the sensitivity peak is defined in (3.2). The sensitivity peak is studied in terms of \(k_2\) and an example is depicted in Figure 3.16 for a controller-gain \(k_1 = 17 \text{ V/rad}\), \(\tau = 0\) and a sampling-time \(h = 2 \text{ ms}\). This shows that the influence of \(k_2\) is similar to the state-feedback case. The difference with the state-feedback case is that the values of \(S_m\) have increased and that for the value of \(k_2 = 0.11 \text{ Vs/rad}\) the smallest value of \(S_m\) appears. This is a relatively small decrease compared to the state-feedback case.

Similar to the state-feedback case, the influence of \(k_1\), \(h\) and \(\tau\) on the stability peak is studied. First, the influence of \(k_1\) is investigated by studying the values of \(S_m\) in terms of \(k_2\) for several values of \(k_1\), \(\tau = 0\) and a sampling-time \(h = 2 \text{ ms}\). The result is depicted in Figure 3.17 where the solid line corresponds to Figure 3.16. Corresponding to the state-feedback case, the value of \(S_m\) is larger when the stability bounds, as derived in Paragraph 3.1.2 are approached. For the output-feedback case, the sensitivity peak increases if the value of \(k_1\) is larger. Different to the state-feedback case, this holds for all values of \(k_2\). The influence of \(k_1\) on the sensitivity peak itself has increased, compared to the state-feedback case.

To study the influence of \(h\), the obtained sensitivity peak in terms of \(k_2\) is depicted in Figure 3.18.
for several values of $h$, $\tau = 0$ and a controller-gain $k_1 = 17$ V/rad. Again, the solid line corresponds to Figure 3.16. The influence of $h$ on the largest values of $S_m$ corresponds to the results on the stability obtained in Paragraph 3.1.2. Similar to the state-feedback case, the sensitivity peak increases if the sampling-time $h$ increases. This influence is larger than the influence of $h$ on the state-feedback controlled PATO set-up.

Besides the influence of $k_1$ and $h$, the influence of the constant time-delay $\tau$ on the robustness of the stability is studied. Therefore, the value of $S_m$, in terms of $k_2$, is depicted in Figure 3.19 for several values of $\frac{\tau}{h}$, a controller-gain $k_1 = 17$ V/rad and a sampling-time $h = 2$ ms. Here, the solid line corresponds to Figure 3.16 and the influence of the delay on the largest values of $S_m$ corresponds to the stability analysis in Paragraph 3.1.2. Figure 3.19 shows that the amplitude of the sensitivity peak increases if the delay increases. Except for the value of $\frac{\tau}{h} = 0.25$, this corresponds to the state-feedback case. Another difference to the state-feedback case is that the influence of the delay is larger.

Again, the demanded stability robustness of the PATO set-up is studied. Therefore, Figure 3.20 depicts all values of $k_2$ which guarantee stability but where $S_m > 2$ in terms of a constant delay $\tau$.

Figure 3.18: The maximum magnitude of the sensitivity $S_m$ in terms of controller-gain $k_2$ for several sampling-times $h$, no time-delay and a controller-gain $k_1 = 17$ V/rad.

Figure 3.19: The maximum magnitude of the sensitivity $S_m$ in terms of controller-gain $k_2$ for several constant time-delays $\frac{\tau}{h}$, a controller-gain $k_1 = 17$ V/rad and a constant sampling-time $h = 2$ ms.

Figure 3.20: The area that guarantees stability, but where the stability robustness criterium is not met ($S_m > 2$) in terms of controller-gain $k_2$ and constant time-delay $\frac{\tau}{h}$ for a controller $k_1 = 17$ V/rad and a sampling-time $h = 2$ ms.
depicted as a fraction of $h$, for the controller-gain $k_1 = 17 \text{ V/rad}$ and the sampling-time $h = 2 \text{ ms}$. The solid line corresponds to the minimum and maximum stabilizing values of $k_2$, as determined in Section 3.1. From Figure 3.20 it is concluded that there are no controller-gains $k_2$ that satisfy the demanded level of stability robustness. To solve this, the value of $k_1$ and/or the value of $h$ can be decreased. This results in a decrease of the sensitivity peak as seen in Figures 3.17 and 3.18.

3.3 Discussion

In the previous sections, the stability and the robustness of the stability are investigated for the PATO set-up with the state- and the output-feedback controller, depending on different controller-gains, sampling-times and constant time-delays. For the state-feedback case, it is shown that time-delays can have a stabilizing effect. This was denoted by the stability peak. For both cases it is shown that an increased value of $k_1$ and/or $h$ causes a decrease of the stability region and a decrease of the robustness of the stability to parameter changes. For the state-feedback case, more stabilizing values of $k_2$ can be chosen if a time-delay is implemented. The shape of the stability region returns in the region that satisfies the demanded robustness of the stability. If an output-feedback controller is implemented for $k_1 = 17 \text{ V/rad}$ and $h = 2 \text{ ms}$, no controller-gains $k_2$ can be found that satisfy a sufficient level of stability robustness. To decrease the amplitude of the sensitivity peak, the value of $k_1$ and/or the value of $h$ can be decreased.
Chapter 4

Performance analysis for constant time-delays

Based on the discrete-time NCS model of Chapter 2, the performance of the PATO set-up is investigated in terms of the settling-time and the bandwidth. In this chapter, the settling-time $t_s$ is defined as the time it takes the time-response of the system to decay to an error-bound around zero and stays within this bound. The bandwidth $f_b$ relates to the maximum frequency at which the output of a system will track an input sinusoid. The influence of a different choice of the controller-gains $k_1$ and $k_2$, the sampling-time $h$ and the constant time-delay $\tau$ on these performance characteristics are investigated. In Section 4.1 the performance in terms of the settling-time is described. Section 4.2 discusses the performance in terms of the bandwidth. In Section 4.3 the different performance criteria and their results are discussed. In each section, both the state-feedback and the output-feedback case are described.

4.1 Settling-time

To determine the effect of the controller-gain $k_2$ on the settling-time, the maximum absolute value of the eigenvalues, denoted by the largest eigenvalue, is studied because it is related to the settling-time [2]. This relationship is in such a way that a smaller settling-time corresponds to a smaller value of the largest eigenvalue. Mathematically, the largest eigenvalue is given by:

$$\bar{\lambda} = \max_{i=1,2,...,q} |\lambda_i|,$$

with $\lambda_i, i = 1,2,...,q$, the $i^{th}$ eigenvalue of the closed-loop, discrete-time NCS model and $q$ the dimension of the system matrix $\Psi$. For the state-feedback case with small time-delays, i.e. $\tau < h$, the closed-loop, discrete-time PATO model in (2.7), (2.8), (2.16), (2.18) is used. For the output-feedback case, with small delays, the closed-loop, discrete-time model in (2.24), (2.25) is used. The smallest value of $\bar{\lambda}$ for all stabilizing values of $k_2$ is denoted as the fastest eigenvalue $\lambda_s$. The value of $k_2$ where $\lambda_s$ appears is denoted as $k_{2,s}$. To obtain the settling-time of the simulated time-response, an error-bound on the angular position error is considered. The error-bound is chosen very small and is given by $\epsilon_x = \pm 10^{-7}$ rad. At the end of this section, the settling-time of the PATO set-up is desired to be smaller than 100 ms. This can be any value, but $t_s \leq 100$ ms is chosen as an example.
4.1 Settling-time

4.1.1 State-feedback control

For the state-feedback case, the closed-loop, discrete-time system (2.7), (2.8), (2.16), (2.18), $C = I$, i.e. all states are measured, for small delays ($\tau < h$) is considered. First, the largest eigenvalue $\lambda$ is determined in terms of $k_2$ for the controller-gain $k_1 = 17$ V/rad, no time-delay and the sampling-time $h = 2$ ms. The results are represented by the solid line in Figure 4.1. Here, the dotted line shows the fastest eigenvalue $\lambda_s$ at point B with the corresponding controller-gain $k_{2,s} = 0.16$ Vs/rad. Note that the eigenvalues in the gray area indicate an unstable system, because they are located outside the unit-circle. The points A and D in Figure 4.1 correspond to the minimum and maximum stabilizing values of $k_2$ derived in Paragraph 3.1.1. The abrupt change of the largest eigenvalues at the points B and C is caused by the fact that a different eigenvalue becomes dominant in terms of its absolute value. To show that the fastest eigenvalue corresponds to the fastest time-response [2], the simulated time-responses for several values of $k_2$ are depicted in Figure 4.2. Here, the solid line, which represents the time-response at point B, has the smallest settling-time and, indeed, $\lambda_s$ corresponds to the smallest settling-time.

To study the influence of $k_1$, several values are chosen. The obtained values of $\bar{\lambda}$ of the system matrix in (2.8), (2.16), (2.18) in terms of $k_2$ are depicted in Figure 4.3 for $\tau = 0$ and $h = 2$ ms. Here, the solid line corresponds to Figure 4.1. Analogous to Paragraph 3.1.1, it is seen that the smallest value of $k_2$, for which the system (2.7), (2.8), (2.16), (2.18) is stabilized (i.e. $\bar{\lambda} < 1$) depends on $k_1$. Moreover, Figure 4.3 shows that the optimum value of $\bar{\lambda}$ (i.e. $\lambda_s$) decreases for increasing values of $k_1$. Note that the related value of $k_2$ (i.e. $k_{2,s}$) increases as well. Obviously, due to the relationship between $\bar{\lambda}$ and the settling-time, a faster response is obtained for increasing values of $k_1$, if $k_2 \geq k_{2,s}$. This is depicted in Figure 4.4 for system (2.7), (2.8), (2.16), (2.18), with $k_2 = 0.16$ Vs/rad, $\tau = 0$ and $h = 2$ ms.

Besides the influence of $k_1$, the influence of the sampling-time $h$ on the settling-time is investigated based on the value of $\bar{\lambda}$ of the system matrix in (2.8), (2.16), (2.18). Some examples for several values of $h$ and $k_2$, with $\tau = 0$ and $k_1 = 17$ V/rad, are depicted in Figure 4.5. Again, the solid line corresponds to Figure 4.1. Similar to Paragraph 3.1.1, Figure 4.5 shows that an increase of $h$ leads to a smaller region of stabilizing values of $k_2$. Moreover, it shows that if the sampling-time decreases, the largest eigenvalue $\lambda$ increases, except for values of $k_2$ near the stability bounds. This means that, in general, the settling-time increases if the value of $h$ decreases. This is also concluded from Figure 4.6 which depicts the simulated time-responses for several values of $h$, with $k_1 = 17$ V/rad, $k_2 = 0.15$ Vs/rad and $\tau = 0$.

Next, the influence of the constant time-delay $\tau$ on the settling-time is investigated. In Figure 4.7

Figure 4.1: The largest eigenvalue $\bar{\lambda}$ in terms of $k_2$ for a controller-gain $k_1 = 17$ V/rad, no time-delay and a sampling-time $h = 2$ ms.

Figure 4.2: The simulated time-responses for several values of $k_2$, a controller-gain $k_1 = 17$ V/rad, no time-delay and a sampling-time $h = 2$ ms.
the obtained values of $\bar{\lambda}$ of the system matrix in (2.9), (2.10), (2.18) are depicted in terms of $k_2$ for several constant time-delays $\tau$, $k_1 = 17 \text{ V/rad}$ and $h = 2 \text{ ms}$. Note that the delay is depicted as a fraction of the sampling-time, i.e. $\frac{\tau}{h}$. In Figure 4.1, the solid line corresponds to Figure 4.1. Analogous to Paragraph 3.1.1, an increase up to $\frac{\tau}{h} = 0.25$ leads to an increase of the number of stabilizing values of $k_2$, while for $\frac{\tau}{h} > 0.25$ an increase of the delay results in a decrease of the number of stabilizing values of $k_2$, as depicted in Figure 4.7(b). From Figure 4.7(a) (i.e. $\frac{\tau}{h} \leq 0.25$), it is clear that the influence of the time-delay on the largest eigenvalue is negligible, except for values of $k_2$ near $k_{2,s}$, i.e. the optimal value of $k_2$ as defined in Figure 4.1. Then, the largest eigenvalue decreases if the delay increases. From Figure 4.7(b) (i.e. $\frac{\tau}{h} > 0.25$), it is clear that the largest eigenvalue increases as the time-delay increases. In Figure 4.7, it is concluded that, analogous to the effect that causes the stability peak (see Paragraph 3.1.1), first, the delay has a positive influence on the largest eigenvalue (if the delay increases for $\frac{\tau}{h} \leq 0.2$) and, second, the delay has a negative influence on the largest eigenvalue (if the delay increases for $0.2 < \frac{\tau}{h}$). This means that the fastest eigenvalue, i.e. the optimal value of $\bar{\lambda}$, appears for a delay of $\frac{\tau}{h} = 0.2$, which is a bit smaller than the value of $\tau$ for which the stability peak appears. Due to the relationship between $\bar{\lambda}$ and the settling-time, a faster response is obtained for $\frac{\tau}{h} = 0.2$. This is depicted in Figure 4.8 for system (2.7), (2.8), (2.16), (2.18), with $k_1 = 17 \text{ V/rad}$, $k_2 = 0.145 \text{ Vs/rad}$ and $h = 2$.
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Figure 4.7: The largest eigenvalue $\lambda$ in terms of $k_2$ for several values of constant time-delay $\tau$, a controller-gain $k_1 = 17$ V/rad and a sampling-time $h = 2$ ms.

Next, the desired maximum settling-time is taken into account. The gray area in Figure 4.10 depicts all values of $k_2$ that guarantee stability but where the desired maximum settling-time is not met, i.e. $t_s > 100$ ms, in terms of the constant time-delay $\tau$ for $k_1 = 17$ V/rad and $h = 2$ ms. Recall that the delay is depicted as a fraction of the sampling-time. The solid line represents the minimum and maximum stabilizing values of $k_2$, as obtained in Paragraph 3.1.1. The dashed line represents the value of $k_2$ for which the fastest eigenvalue $\lambda_s$, and thus the smallest settling-time, appears. From Figure 4.7 it is concluded that the effect that leads to the stability peak, results in an increase of the possible values of $k_2$ that satisfy the desired settling-time. Moreover, it is concluded that the smallest settling-time appears around $k_{2,s} = 0.15$ Vs/rad and that the influence of the constant time-delay on this value is relatively small.

Finally, the influence of the constant delay on the settling-time is determined by studying the simulated settling-times for several constant delays $\tau$ (depicted as a fraction of the sampling-time), $k_1 = 17$ V/rad, $k_2 = k_{2,s}(\tau)$ and $h = 2$ ms is studied. The result is depicted in Figure 4.10 which

Figure 4.8: The simulated time-responses for several values of constant time-delay $\tau$, controller-gains $k_1 = 17$ V/rad and $k_2 = 0.145$ Vs/rad and a sampling-time $h = 2$ ms.
shows that, indeed, the settling-time decreases if the delay increases for \( \tau_h \leq 0.2 \) and increases if the delay increases for \( \tau_h > 0.2 \). Therefore, it is concluded that, for \( k_1 = 17 \) V/rad and \( h = 2 \) ms, the smallest settling-time appears for \( k_2 = 0.14 \) Vs/rad and \( \tau_h = 0.2 \).

### 4.1.2 Output-feedback control

For the output-feedback case, the closed-loop, discrete-time system (2.24), (2.25) is considered. First, the effect of \( k_2 \) on the settling-time of this system is determined based on the largest eigenvalue \( \lambda \) of the system matrix \( \Psi \) in (2.25). The obtained values of \( \lambda \), in terms of \( k_2 \) with \( \tau = 0 \), \( k_1 = 17 \) V/rad and \( h = 2 \) ms, are depicted by the solid line in Figure 4.11. The dotted line shows the optimal value of \( \lambda \), denoted by fastest eigenvalue \( \lambda_s \), with the corresponding controller-gain \( k_{2,s} \), given by point \( B \). Recall that the eigenvalues in the gray area indicate an unstable system. The shape of the solid line is comparable to that of the state-feedback case, see Figure 4.1. The minimum and maximum stabilizing value of \( k_2 \), given by points \( A \) and \( D \), correspond to the stability bounds in Paragraph 3.1.2. To verify that the fastest response appears at point \( B \), with \( k_{2,s} = 0.12 \) Vs/rad, simulated time-responses for different values of \( k_2 \) are depicted in Figure 4.12.
This shows that the solid line, which corresponds to point B, has indeed the smallest settling-time.

To study the influence of $k_1$, several values are chosen. The obtained values of $\lambda$ of the system matrix $\Psi$ in (2.25) in terms of $k_2$ are depicted in Figure 4.13 for $\tau = 0$ and $h = 2$ ms. Here, the solid line corresponds to Figure 4.11. Analogous to Paragraph 3.1.2, it is seen that both the smallest and the largest value of $k_2$ for which the system (2.24), (2.25) is stabilized (i.e. $\lambda < 1$) depend on $k_1$. Moreover, Figure 4.13 shows that the influence of $k_1$ on the optimum value of $\lambda$ (i.e. $\lambda_s$) differs from the state-feedback case (see Figure 4.9). The difference is that, for the output-feedback case, the fastest eigenvalue $\lambda_s$ has a larger value for $k_1 = 15$ V/rad and $k_1 = 17$ V/rad, compared to the fastest eigenvalue for $k_1 = 10$ V/rad, while for the state-feedback case the value of $\lambda_s$ is smaller for a larger value of $k_1$. Note that the related value of $k_2$ (i.e. $k_{2_s}$) increases for increasing values of $k_1$, which corresponds to the state-feedback case. Similar to the state-feedback case, the largest eigenvalue $\lambda$ decreases if the value of $k_1$ increases for $k_2 \geq k_{2_s}$. Obviously, due to the relationship between $\lambda$ and the settling-time, a faster response is obtained for increasing values of $k_1$, for $k_2 \geq k_{2_s}$. This is depicted in Figure 4.14 for system (2.24), (2.25), with $k_2 = 0.12$ Vs/rad, $\tau = 0$ and $h = 2$ ms.

Besides the influence of $k_1$, the influence of the sampling-time $h$ on the settling-time is investigated based on the value of $\lambda$ of the system matrix in (2.25). Some examples for several values of $h$ and $k_2$, with $\tau = 0$ and $k_1 = 17$ V/rad, are depicted in Figure 4.15. Again, the solid line corresponds to Figure 4.11. Similar to Paragraph 3.1.2, Figure 4.15 shows that an increase of $h$ leads to a smaller region of stabilizing values of $k_2$. Moreover, it shows that the influence of $h$ on the largest eigenvalue $\lambda$ is hard to describe, because of the different shapes for $h = 2$ ms and $h = 1.5$ ms. The influence of $h$ on the fastest eigenvalue is such that the value of $\lambda_s$, i.e. the optimal value of $\lambda$, decreases if the value of $h$ decreases. This is a remarkable difference with the state-feedback case, where the value of the fastest eigenvalue (i.e. $\lambda_1$) increases if the sampling-time decreases. The difference is caused by the velocity estimation in the output-feedback controller, which depends on $h$. The influence of $h$ on the value of $k_{2,s}$, corresponding to the fastest eigenvalue, is relatively small. To show this, the simulated time-responses for several values of $h$, with $k_1 = 17$ V/rad, $k_2 = 0.13$ Vs/rad and $\tau = 0$, are depicted in Figure 4.16. This shows that, for this $k_2$-value, the settling-time decreases if the value of $h$ decreases.

Next, the influence of the constant time-delay $\tau$ on the settling-time is studied. The obtained values of $\lambda$ of the system matrix in (2.25) are depicted in Figure 4.17 in terms of $k_2$ for different values of $\tau$, with $k_1 = 17$ V/rad and $h = 2$ ms. Recall that the delay is depicted as a fraction of the sampling-time and that, again, the solid line corresponds to Figure 4.11. Analogous to Paragraph 3.1.2, an increase of the delay does not lead to an increase of the number of stabilizing
values of $k_2$, which was the case for the PATO set-up with a state-feedback controller. Moreover, Figure 4.17 shows that the value of $\overline{\lambda}$ increases if the time-delay increases, which differs from the state-feedback case. For the state-feedback case, the positive influence of the delay on the settling-time is caused by the same effect that resulted in the stability peak. Therefore, it makes sense that time-delays degrade the value of $\overline{\lambda}$, for the output-feedback case. Due to this relationship between $\overline{\lambda}$ and the settling-time, a slower response is obtained if the delay increases. This is depicted in Figure 4.18 for system (2.24), (2.25), with $k_1 = 17$ V/rad, $k_2 = 0.13$ Vs/rad and $h = 2$ ms. Note that, for a clear view, the time-response for $\tau_h = 0.75$ is not shown in the zoom.

Similar to the state-feedback case, the demands on the settling-time of the PATO set-up need to be included. In Figure 4.19, the gray area depicts all values of $k_2$ for $k_1 = 17$ V/rad and $h = 2$ ms that guarantee stability, however the demands on the settling-time are not met (i.e. $t_s > 100$ ms). The solid line corresponds to the stability bounds, as determined in Paragraph 3.1.2. The dashed line represents the value of $k_{2,s}$, where the fastest eigenvalue, i.e. the optimal value of $\overline{\lambda}$ (and thus the settling-time), appears for different values of $\overline{\xi}$. From Figure 4.19 it is concluded that the possible controller settings, with respect to the settling-time, have decreased, compared to the stability bounds. This decrease is larger than the decrease for the state-feedback case. It is also concluded that the smallest settling-time appears around $k_{2,s} = 0.13$ Vs/rad for $\tau = 0$, which
is a bit smaller than for the state-feedback case. Similar to the state-feedback case, the influence of the delay on the value of $k_{2,s}$ is very small.

Finally, the influence of the constant delay on the settling-time is determined by studying the simulated settling-times for several constant delays $\tau$ (depicted as a fraction of the sampling-time), $k_1 = 17 \text{ V/rad}$, $k_2 = k_{2,s}(\tau)$ and $h = 2 \text{ ms}$ is studied. The result is depicted in Figure 4.20, which shows that the settling-time increases if the delay increases. Therefore, it is concluded that, for $k_1 = 17 \text{ V/rad}$ and $h = 2 \text{ ms}$, the smallest settling-time appears for $k_2 = 0.12 \text{ Vs/rad}$ and no time-delay.

4.2 Bandwidth

In this section, the bandwidth of the controlled system is determined by studying the open-loop frequency-response function of the transfer function $L(z)$ between the system error $E(z)$ and the system output $Y(z)$. The bandwidth is denoted as the frequency for which the magnitude of the stable open-loop response crosses 1 (=0 dB). The bandwidth is determined in terms of the controller-gain $k_2$. The influence of the controller-gain $k_1$, the sampling-time $h$ and the constant delay $\tau$ on the bandwidth is also analyzed. At the end of this section, the maximum allowable bandwidth $f_b$ is set to 100 Hz to prevent the effect of aliasing [2]. This value is chosen, because a signal must be bandlimited to half the sampling frequency, i.e. $\frac{1}{2}F_s$. To be sure, a maximum bandwidth of $\frac{1}{5}F_s$ is chosen, instead of $\frac{1}{2}F_s$. If the sampling frequency $F_s = 500 \text{ Hz}$ is used, the maximum allowable bandwidth is $f_b = 100 \text{ Hz}$. The bandwidth of the PATO set-up will be studied in Paragraph 4.2.1 for the state-feedback case and in Paragraph 4.2.2 for the output-feedback case.

4.2.1 State-feedback control

To simplify the computation for the state-feedback case, the estimated open-loop frequency-response function in (2.20), (2.21) is considered, instead of the open-loop frequency-response function in (2.20), (2.19). Here, $L(z)$ describes the transfer function between the system error $E(z)$ and the system output $Y(z)$. Recall that the equivalent characteristics in the $z$-domain are given by the expression $z = e^{j2\pi f h}$, with $f$ the frequency and $h$ the sampling-time. To determine the bandwidth of the system, open-loop frequency-response functions are studied. An example is presented in Figure 4.21 where the solid line represents the open-loop frequency-response function.
for the controller-gains $k_1 = 17 \text{ V/rad}$ and $k_2 = 0.15 \text{ Vs/rad}$, no time-delay and the sampling-time $h = 2 \text{ ms}$. The dotted line corresponds to the bandwidth. Recall that, due to the diagonal structure, the system in (2.21) is studied based on two SISO models. Therefore, the bandwidths $f_{b,1}$ and $f_{b,2}$ are of interest. The bandwidth is determined by the smallest value of $f_{b,1}$ and $f_{b,2}$, i.e. $f_b = \min\{f_{b,1}, f_{b,2}\}$, because for this value it is guaranteed that the output of the system will track an input sinusoid with this frequency. From Figure 4.21 it follows that $f_{b,1} = 48.5 \text{ Hz}$ and $f_{b,2} = 50.6 \text{ Hz}$, which shows that $f_b = 48.5$. Not that this value satisfies the demand on the bandwidth $f_b \leq 100 \text{ Hz}$.

First, the influence of $k_2$ on the bandwidth of the open-loop frequency-response function in (2.21) is studied in terms of $k_2$, with the controller-gain $k_1 = 17 \text{ V/rad}$, no time-delay and the sampling-time $h = 2 \text{ ms}$. The results are depicted in Figure 4.22. Here, the points A and B correspond to either the minimum stabilizing value of $k_2$ or the maximum stabilizing value of $k_2$, derived in Paragraph 3.1.1. Note that, to ensure stability only the range of $k_2 \in [k_{2,1}, k_{2,2}]$ is used. Because the frequency-response function in (2.20), (2.21) is an estimation of the exact closed-loop discrete-time model in (2.7), (2.8), (2.16), (2.18), there is a relatively small difference between the points.

**Figure 4.21:** The open-loop frequency-response function and the corresponding bandwidth for the controller-gains $k_1 = 17 \text{ V/rad}$ and $k_2 = 0.15 \text{ Vs/rad}$, no time-delay and the sampling-time $h = 2 \text{ ms}$. 

**Figure 4.22:** The bandwidth in terms of controller-gain $k_2$ for the controller-gain $k_1 = 17 \text{ V/rad}$, no time-delay and the sampling-time $h = 2 \text{ ms}$. 

**Figure 4.23:** The bandwidth in terms of controller-gain $k_2$ for several values of $k_1$, no time-delay and the sampling-time $h = 2 \text{ ms}$.
Figure 4.24: The bandwidth in terms of controller-gain \( k_2 \) for several values of \( h \), the controller-gain \( k_1 = 17 \) V/rad and no time-delay.

Figure 4.25: The bandwidth in terms of controller-gain \( k_2 \) for several delays \( \tau \), the controller-gain \( k_1 = 17 \) V/rad and the sampling-time \( h = 2 \) ms.

A and B in Figure 4.22 and the minimum and maximum stabilizing values of \( k_2 \), as determined in Paragraph 3.1.1. Moreover, Figure 4.22 shows that the bandwidth increases if the value of \( k_2 \) increases. This is expected because an increase of the value of \( k_2 \) results in an increase of the magnitude of the open-loop response for higher frequencies.

Next, several values of \( k_1 \) are chosen to study the influence of \( k_1 \) on the bandwidth of the open-loop frequency-response function (2.20), (2.21). The obtained bandwidth is depicted in Figure 4.23 in terms of \( k_2 \), with no time-delay and the sampling-time \( h = 2 \) ms, where the solid line corresponds to Figure 4.22 and the range of \( k_2 \in [k_2, \bar{k}_2] \) is used to ensure stability. Analogous to Paragraph 3.1.1 it is seen that the smallest value of \( k_3 \), for which the system (2.7), (2.8), (2.16), (2.18) is stabilized depends on \( k_1 \). Different to Paragraph 3.1.1 it is seen that the largest stabilizing value of \( k_2 \) depends on \( k_1 \). This difference is caused by the forward Euler substitution rule, given in discrete-time by \( e_{v,k} = e_{x,k+1} - e_{x,k} h \), which is used in the frequency-response function (2.20), (2.21) to estimate the angular velocity error in terms of the angular position error and vice versa. Due to this substitution rule, the transfer function between the angular velocity error and the angular velocity output is dependent on \( k_1 \). Moreover, Figure 4.23 shows that the bandwidth increases if the value of \( k_1 \) increases for \( k_2 \leq 0.14 \) V/s/rad and that the bandwidth decreases if the value of \( k_1 \) increases for values of \( k_2 > 0.14 \) V/s/rad.

Figure 4.26: The area that guarantees an upper-bound on the bandwidth in terms of controller-gain \( k_2 \) and the constant time-delay \( \tau \) for the controller-gain \( k_1 = 17 \) V/rad and the sampling-time \( h = 2 \) ms.
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Next, the influence of \( h \) on the bandwidth of the open-loop frequency-response function in (2.20), (2.21) is investigated. Therefore, the obtained results for different values of \( h \) and \( k_2 \), with \( k_1 = 17 \text{ V/rad} \) and no time-delay, are depicted in Figure 4.24. Again, the solid line corresponds to Figure 4.22. Similar to Paragraph 3.1.1, Figure 4.24 shows that an increase of \( h \) leads to a smaller region of stabilizing values of \( k_2 \). Moreover, Figure 4.24 shows that the bandwidth increases if the sampling-time decreases. The influence of \( h \) on the bandwidth increases if the value of \( k_2 \) increases.

Besides the influence of \( k_1 \) and \( h \), the influence of the constant time-delay \( \tau \) on the bandwidth is studied. The obtained bandwidth of the open-loop frequency-response function in (2.20), (2.21) for different values of \( k_2 \) and \( \tau \), with \( k_1 = 17 \text{ V/rad} \) and \( h = 2 \text{ ms} \), is depicted in Figure 4.25, where the dashed line corresponds to Figure 4.22. Note that the delay is shown as a fraction of the sampling-time, i.e. \( \frac{\tau}{h} \). From Figure 4.25 it is concluded that, besides the influence of the delay on the stability bounds, the delay does not affect the bandwidth. To show this effect, Figure 4.27 gives the open-loop frequency-response functions for several constant time-delays \( \tau \), the controller-gains \( k_1 = 17 \text{ V/rad} \) and \( k_2 = 0.3 \text{ Vs/rad} \) and the sampling-time \( h = 2 \text{ ms} \). Similar to Figure 4.21, the transfer function \( \tilde{L}_1(z) \) (see Figure 4.27(a)) determines the bandwidth of the controlled system. From Figure 4.27 it is concluded that the delay only affects the magnitude of the open-loop response for frequencies larger than the bandwidth, i.e. \( f > f_b \).

Finally, the demanded bandwidth is taken into account. The gray area in Figure 4.26 depicts all values of \( k_2 \) that guarantee stability but where the demanded bandwidth is not met, i.e. \( f_b > 100 \text{ Hz} \), in terms of a constant delay \( \tau \), shown as a fraction of the sampling-time, with \( k_1 = 17 \text{ V/rad} \) and \( h = 2 \text{ ms} \). Recall that the solid line corresponds to the minimum and maximum stabilizing values of \( k_2 \), see Paragraph 3.1.1. From Figure 4.26 it is concluded that the value of \( k_2 \) should be smaller than approximately 0.34 \text{ Vs/rad}. Combined with the observations in Figure 4.25 it is concluded that the effect that causes the stability peak, does not influence the bandwidth.

4.2.2 Output-feedback control

To study the bandwidth of the PATO set-up with an output-feedback controller, the open-loop frequency-response function in (2.20), (2.29) is considered. Here, \( L(z) \) describes the transfer function between the angular position error \( E_x(z) \) and the angular position output \( Y_x(z) \).

First, the influence of \( k_2 \) on the bandwidth of the open-loop frequency-response function in (2.20), (2.29) is studied for the controller-gain \( k_1 = 17 \text{ V/rad}, \tau = 0 \) and \( h = 2 \text{ ms} \). The results are
depicted in Figure 4.28 where the points A and B correspond to the minimum and maximum stabilizing values of $k_2$, respectively, as derived in Paragraph 3.1.2. Recall that the range of $k_2 \in [k_2, k_2]$ is used to ensure stability. Similar to the state-feedback case (see Figure 4.24), the bandwidth is larger for a larger value of $k_2$. To determine whether or not the bandwidth is changed compared to the state-feedback case, the bandwidth is determined for several values of $k_2$. For the state-feedback case, the bandwidth for the value of $k_2 = 0.1$ V/s/rad is $f_b = 37$ Hz and $f_b = 110$ Hz for $k_2 = 0.4$ V/s/rad. For the output-feedback case, the bandwidth for the value of $k_2 = 0.1$ V/s/rad is $f_b = 44$ Hz and $f_b = 117$ Hz for $k_2 = 0.4$ V/s/rad. Therefore, it is concluded that the bandwidth is larger for the output-feedback case. The difference between the state- and output-feedback controller is caused by the Euler backward function that is used in the output-feedback controller.

To study the influence of $k_1$ on the bandwidth of the open-loop frequency-response function in (2.20), (2.29), several values of $k_1$ are chosen. The obtained bandwidth, in terms of $k_2$, is depicted in Figure 4.29 for $\tau = 0$ and $h = 2$ ms. Here, the solid line corresponds to Figure 4.28 and, again, the range of $k_2 \in [k_2, k_2]$ is used to ensure stability. Analogous to Paragraph 3.1.2, it is seen that both the smallest and the largest value of $k_2$, for which the system (2.24), (2.25) is stabilized depend on $k_1$. Moreover, Figure 4.29 shows that the bandwidth increases as the value of $k_1$ increases. This corresponds to the state-feedback case.
Figure 4.32: The area that guarantees an upper-bound on the bandwidth in terms of controller-gain $k_2$ and the constant time-delay $\tau$ for the controller-gain $k_1 = 17$ V/rad and the sampling-time $h = 2$ ms.

Moreover, the influence of $h$ on the bandwidth is studied. The obtained bandwidth of the open-loop frequency-response function in (2.29), (2.30), in terms of $k_2$, is depicted in Figure 4.30 for several values of $h$, with $k_1 = 17$ V/rad and $\tau = 0$. Again, the solid line corresponds to Figure 4.28. Similar to Paragraph 3.1.2, Figure 4.24 shows that an increase of $h$ leads to a smaller region of stabilizing values of $k_2$. Moreover, Figure 4.30 shows that if the sampling-time decreases, the bandwidth decreases for small values of $k_2$ (i.e. $k_2 < 0.14$ Vs/rad) and increases for large values of $k_2$ (i.e. $k_2 > 0.14$ Vs/rad). This differs from the state-feedback case, where the bandwidth decreases if the sampling-time increases. The difference is caused by the Euler backward estimation of the angular velocity error that influences the magnitude of the controller.

Besides the influence of $k_1$ and $h$, the influence of the constant time-delay $\tau$ on the bandwidth is studied, by determining the bandwidth of the open-loop frequency-response function in (2.29) for different values of $\tau$ and $k_2$, with $k_1 = 17$ V/rad and $h = 2$ ms. The result is depicted in Figure 4.31, where the delay is shown as a fraction of the sampling-time. Here, the dashed line corresponds to Figure 4.28. From Figure 4.31 it is concluded that, compared to the state-feedback case, the influence of $\tau$ on the bandwidth has not changed, namely that the bandwidth is not affected by the delay.

Similar to the state-feedback case, the demanded bandwidth of the PATO set-up is taken into account. Therefore, the gray area in Figure 4.32 shows all values of $k_2$ that guarantee stability but where the bandwidth $f_b > 100$ Hz in terms of $\frac{k}{h}$ with $k_1 = 17$ V/rad and $h = 2$ ms. The solid line corresponds to the stability bounds, as determined in Paragraph 3.1.2. From Figure 4.32 it is concluded that if $k_2 \leq 0.315$ Vs/rad the demands on the bandwidth are met. Clearly, the influence of the delay on this value of $k_2$, where the maximum allowed bandwidth appears, is negligible. Note that, compared to the state-feedback case (see Figure 4.26), the area where the system is stable, but does not satisfy the demand on the bandwidth is much smaller.

4.3 Discussion

In the previous sections, the settling-time and the bandwidth are investigated for the PATO set-up with the state- and the output-feedback controller. Their dependence on different controller-gains, sampling-times and constant time-delays is studied. For the state-feedback case, it is shown that, for most stabilizing controllers, small time-delays may decrease the settling-time. However, the bandwidth is not influenced by an increase of the constant delay. For large delays, an increase of the delay may lead to an increase of the settling-time. Again, the bandwidth is not influenced by
the delay. For all delays, it is shown that the largest bandwidth is not achievable if the smallest settling-time is demanded, which makes a trade-off between these performance criteria necessary. For the output-feedback case, the same trade-off between the settling-time and the bandwidth holds. Note that here an increase of the delays always results in an increase of the settling-time and again the bandwidth is not affected.

For both cases, the smallest settling-time can be found by performing a model-based eigenvalue analysis and by studying simulations, because the smallest settling-time matches with the smallest maximum absolute value of the closed-loop eigenvalues. For the studied second-order system, the settling-time decreases for larger controller-gains until a transition point is met, then the settling-time is larger for larger controller-gains. The settling-time of the studied closed-loop system also increases if the sampling-time is increased, but this may destabilize the closed-loop system. For the bandwidth, it is shown that it is not affected by constant time-delays. The optimal bandwidth, i.e. the largest bandwidth, appears for the maximum stabilizing controller-gains. For both controller types, a larger bandwidth is obtained by increasing the value of the controller-gains and/or by decreasing the value of the sampling-time.

If the results on the criteria on the sensitivity, i.e. $S_m \leq 2$ (see Section 3.2), the settling-time, i.e. $t_s \leq 100$ ms, and the bandwidth, i.e. $f_b \leq 100$ Hz, are compared, it is concluded that for this example the criterium on the stability robustness ($S_m \leq 2$) is the most restrictive to determine the possible controller settings. For the state-feedback case, it is concluded that the occurrence of the stability peak leads to an increase of the possible controller settings and that the best performance, i.e. smallest settling-time and largest robustness for stability, is obtained for $k_2 = 0.15$ Vs/rad, with a constant relative delay $\tau = 0.2$. For the output-feedback case, it is concluded that the value of $k_1$ and/or the value of $h$ needs to be decreased to guarantee that the performance requirements are met. This results in a decrease of the sensitivity peak as seen in Figure 3.17 and in Figure 3.18.
Chapter 5

The PATO set-up

An NCS contains a network which induces unknown, time-varying delays. Therefore, it cannot be used to validate the model-based results on an NCS with constant time-delays, as obtained in Chapter 3 and Chapter 4. The PATO set-up, however, has no network and because constant time-delays are implemented between the controller and the plant, it represents an experimental NCS with constant time-delays, which can be used for the purpose of validation. In this chapter, the PATO set-up is introduced. In Section 5.1 a general description and the properties of the PATO set-up are discussed. In Section 5.2 the implementation of time-delays smaller than the sampling-time is explained. To ensure that only constant time-delays are implemented, in Section 5.3 it is studied if the internal delay is indeed constant and reproducible. In Section 5.4 a general description of the measurement procedure for the time-domain and the frequency-domain measurements is provided.

5.1 The PATO set-up

The experimental set-up is a second-order system (a single actuated inertia) which is used in the DCT lab of the TU Eindhoven and is called the PATO set-up. The PATO set-up is chosen for its simplicity [7] and is depicted in Figure 5.1 with the TU/e DAC system and the amplifier. Originally, the PATO set-up is a twin motor set-up. For the experiments in this report, only one motor, with an inertia attached to it, is used. A schematic representation of the PATO set-up is

![Figure 5.1: A picture of the PATO set-up.](image-url)
5.2 Implementation of constant time-delays smaller than the sampling-time

Measurements on the PATO set-up are performed with Matlab/Simulink. In Matlab/Simulink a delay, indicated in the $z$-domain by $z^{-1}$, is a time-delay equal to the sampling-time. This delay is denoted as a unit-delay. To investigate the influence of time-delays smaller than the sampling-time, i.e. smaller than a unit-delay, the plant is sampled at a sampling frequency of 500 Hz, while the controller computation runs at an operating frequency of 5 kHz. By doing this, a unit-delay of one sample at an operating frequency of 5 kHz corresponds to a delay of 10% of the sampling-time at a sampling frequency of 500 Hz. To implement a sampling and an operating frequency, an adapted zero-order-hold block (Z.O.H.∗) is introduced in the Simulink implementation of the controller.

The adapted zero-order-hold block, as depicted in Figure 5.3, operates, similar to the controller, at the operating frequency of 5 kHz, with $h_o = \frac{1}{5000}$ s. This block is used to sample the plant at a frequency of 500 Hz, with $h = \frac{1}{500}$ s, such that delays smaller than the sampling-time can be included. The zero-order-hold block consists of a number of unit-delays, pulse generators and switches. Each unit-delay results in a time-delay of $\frac{1}{5000}$ s of the current signal. Each pulse
Chapter 5: The PATO set-up

5.3 Validation of the internal delay of the PATO set-up

Internal delay, denoted by $\tau_c$, may appear when measurements are performed on the PATO set-up. This is caused by, e.g., delay in the wiring, delay in the amplifier, delay in the TU/e DAC system and computational delay of the controller. To ensure that constant time-delays are implemented on the PATO set-up, the internal delay needs to be constant. Note that this constant delay is part of the total network-induced-delay, as explained in Chapter 2. To increase the possible settings for the constant delay $\tau$, it is desired that the internal delay $\tau_c$ is small, i.e. $\tau_c \ll h$. Then, the obtained model-based results can be validated for delays close to zero. To determine the internal delay, the open-loop frequency-response of the PATO set-up is measured for controller-gains $k_1 = 17$ V/rad and $k_2 = 0.2$ Vs/rad, no (additional) time-delay, i.e. $\tau = 0$, and a sampling-time $h = 2 \text{ ms}$. 

![Figure 5.3: A schematic representation of the zero-order-hold block.](image)
42 5.4 Performing time- and frequency-response measurements

ms. Note that these values are chosen, because these correspond to the experiments performed in Chapter [9]. The result is depicted in Figure 5.4 with the open-loop frequency-response of several constant delays, given by $e^{-j\omega(\frac{h}{2} + \tau_c)}$ [2], with $h$ the sampling-time and $\tau_c$ the internal delay. Note that the internal delay is depicted as a fraction of the sampling-time, i.e. $\frac{\tau_c}{h}$. Here, it is concluded that the most comparable internal delay is $\tau_c h = 0.02$. To determine whether or not the internal delay is constant, the open-loop response of the PATO set-up is measured several times and for different values of $k_1$ and $k_2$. It is concluded that the internal delay is constant and also relatively small, so constant time-delays can be implemented on the PATO set-up to validate the model-based results on an NCS with constant time-delays as obtained in Chapter [3] and Chapter [4].

5.4 Performing time- and frequency-response measurements

To validate the model-based results, two kinds of measurements are performed. First, the time-response to a reference signal is measured to determine the minimum and maximum stabilizing values of $k_2$ (for fixed $k_1$) and the settling-time for given controller-gains. Second, a frequency-response measurement in closed loop is performed to determine the bandwidth and the sensitivity of the controlled PATO set-up. To limit the influence of the Coulomb friction, a reference signal of the constant angular velocity $r_v = 100 \text{ rad/s}$ is used for all measurements. Then, the reference signal to be tracked is given by $r_{x,k_o} = 100k_o h_o$, with $r_{x,k_o}$ the reference angular position, $k_o$ the sampling instant of the 5 kHz signal and $h_o$ the sampling-time of the 5 kHz signal, given by $h_o = \frac{1}{5000} \text{ s}$. Now the feedforward is given by $u_{ff} = br_v = 0.135 \text{ V}$.

5.4.1 Time-response measurements

Figure 5.5 gives a schematic representation of the time-response measurement set-up, where $k_o$ represents the sampling instant of the operating frequency of 5 kHz. Recall that, effectively, the system operates at the sampling frequency of 500 Hz and that the sampling instant $k = 10k_o$.  

![Figure 5.4](image)

**Figure 5.4:** The measured open-loop frequency response of the PATO set-up compared to several constant delays $\tau_c$ for controller-gains $k_1 = 17 \text{ V/rad}$ and $k_2 = 0.2 \text{ V/s/rad}$, no time-delay, i.e. $\tau = 0$, and a sampling-time $h = 2 \text{ ms}$.
In Figure 5.5, the angular position error is obtained by subtracting the angular position output $y_{x,k_o}$ from the reference angular position $r_{x,k_o}$. The angular position error is down-sampled from the operating frequency of 5 kHz to an effective sampling frequency of 500 Hz, using the adapted zero-order-hold block (Z.O.H.∗). Recall that the properties of the adapted zero-order-hold block are similar to a zero-order-hold that is used in most discrete-time systems. The adapted zero-order-hold block is placed before the output-feedback controller $K$ to ensure that, effectively, the angular position error $e_{x,k_o}$ has a constant sampling-time $h = 2$ ms. The control-input $u_{x,k_o}$ is derived by adding the feedforward to the output of the output-feedback controller $K$, which is defined in [2.23]. In the output-feedback controller $K$, the angular velocity error is estimated with the Euler backward method given by $\hat{e}_{v,k} = \frac{1}{h}(e_{x,k} - e_{x,k-1}) = \frac{1}{h}(e_{x,k_o} - e_{x,k_o-10})$. The derived control-input $u_{k_o}$ is sent over a communication network with a known constant time-delay $\tau$. This communication network is simulated by $N$ unit-delays that represent a constant time-delay of $N \cdot 10\%$ of the sampling-time $h = 2$ ms. Because the angular position error has an effective frequency of 500 Hz and because the feedforward is constant, the control-input $u_{k_o}$ has an effective sampling frequency of 500 Hz. The angular position output $y_{x,k_o}$ is measured by an encoder at the operating frequency of 5 kHz. In Section 5.3, it was concluded that the PATO set-up has an internal delay of 2% of the sampling-time $h$. This is added to the implemented constant time-delay and results in a total constant time-delay $\tau$ is $N \cdot 10\% + 2\%$ of the sampling-time $h$. Note that all measurements will be performed several times to ensure the reproducibility of the results.

### 5.4.2 Frequency-response measurements

Besides the time-response measurement, frequency-response measurements in closed loop are performed. A schematic representation of the measurement set-up is given in Figure 5.6. The only difference with the time-response measurement is that noise $w_{k_o}$ is added to the control-input. To ensure that the noise $w_{k_o}$ has a maximum frequency of 500 Hz, $w_{k_o}$ passes a zero-order-hold block before it is added to the control-input. This results in a perturbed control-input $u_{k_o}$. In this experiment $u_{k_o}$, $w_{k_o}$ and $e_{x,k_o}$ are measured. Similar to the time-response measurements, the total constant time-delay $\tau$ is given as $N \cdot 10\% + 2\%$ of the sampling-time $h$. In Chapter 3 the sensitivity is defined as the closed-loop transfer function from the output disturbances to the system error. Since the disturbance $w_{k_o}$ is added to the control-input instead of to the system output, as in (3.1), the sensitivity function is now defined as the closed-loop transfer function between the noise $w_{k_o}$ and the control-input $u_{k_o}$ and is derived with:

$$U(j2\pi f) = S(j2\pi f)W(j2\pi f),$$

Figure 5.5: A schematic representation of measuring the position error $e_k$ and control-input $u_k$ in time.
with $U(j2\pi f)$ the Fourier transformation of the measured control-input $u_{k_o}$, $W(j2\pi f)$ the Fourier transformation of the measured noise $w_{k_o}$ and $S(j2\pi f)$ the sensitivity function. The open-loop frequency-response function of the PATO set-up is derived with:

$$L(j2\pi f) = S(j2\pi f)^{-1} - 1.$$  

(5.2)

To check the reliability of the measured open-loop response, the coherence between the measured angular position error $\epsilon_{x,k_o}$ and the measured noise $w_{k_o}$ as well as the coherence between the measured control-input $u_{k_o}$ and the measured noise $w_{k_o}$ are derived. Again the reproducibility of the measurements is ensured by repeating measurements several times.
Chapter 6

Experimental validation for constant time-delays

In this chapter, the experimental results for stability and performance analysis of the PATO set-up with constant time-delays are presented and compared to the analytic results obtained in Chapter 3 and Chapter 4. The validation is, due to the fact that the used set-up where only the angular position is measured, limited to the output-feedback case, with the controller defined in (2.22). Section 6.1 presents the measured time-responses and describes whether the measurements show that the experimental closed-loop system is stable or not. The different controller-gains and delay combinations are combined to determine the experimental stability region that can be compared to the analytically obtained stability region of Chapter 3. The performance in terms of the settling-time of the measured time-responses is determined in Section 6.2. The bandwidth and sensitivity, obtained from the frequency-domain measurements, are discussed in Section 6.3 and Section 6.4, respectively. The obtained results are discussed in Section 6.5.

6.1 Stability

In this section, the model-based minimum and maximum stabilizing values of $k_2$, obtained in Chapter 3 and denoted by $k_{2,\text{min}}$ and $k_{2,\text{max}}$, respectively, are validated for several constant time-delays, a controller-gain $k_1 = 15 \text{ V/rad}$ and a sampling-time $h = 2 \text{ ms}$. The stability is determined based on time-domain measurements. A time-response is stable if the measured, discrete-time angular position error $e_{\phi,k}$ converges to zero. Because of the encoder resolution, which induces measurement noise, it is expected that the discrete-time angular position error $e_{\phi,k}$ does not converge to zero completely, but remains within a bound around zero. First, the minimum stabilizing value of $k_2$ is validated, and second, the maximum stabilizing value of $k_2$ is determined.

6.1.1 The minimum stabilizing value of $k_2$

The minimum stabilizing value of $k_2$ is determined for several constant delays, a controller-gain $k_1 = 15 \text{ V/rad}$ and a sampling-time $h = 2 \text{ ms}$. This is performed by increasing the value of $k_2$ for fixed values of $\frac{1}{h}$, $k_1$ and $h$, until the transition between an unstable time-response and a stable time-response is found. An example of a measured time-response is presented in Figure 6.1 for controller-gains $k_1 = 15 \text{ V/rad}$ and $k_2 = 0.02 \text{ V's/rad}$, a constant time-delay $\frac{1}{h} = 0.32$ and a sampling-time $h = 2 \text{ ms}$. Because the reference input is activated at $t = 1 \text{ s}$, the measured time-response for $t \in [0, 1) \text{ s}$ is zero and, therefore, not shown. The measured control-input $u_k$ includes
The measured time-response for controller-gains $k_1 = 15 \text{ V/rad}$ and $k_2 = 0.02 \text{ Vs/rad}$, a constant time-delay $\frac{T}{h} = 0.32$ and a sampling-time $h = 2 \text{ ms}$.

Now, the value of $k_2$ is increased to $k_2 = 0.03 \text{ Vs/rad}$. The corresponding measured time-response is depicted in Figure 6.2 which shows convergence of the measured angular position error $e_{x,k}$ and the measured control-input $u_k$ in time. Notice that there is no exact convergence to zero, which is caused by the measurement noise induced by the encoder resolution and by an inexact feedforward signal, see Chapter 5. Due to saturation of the actuator, the measured control-input
$u_k$ is bounded by $\pm 2.5$ V for $t \in [1, 2.2]$ s. Because the measured angular position error converges to a neighborhood of zero and because the corresponding control-input does not saturate in steady state, it is concluded that the time-response of Figure 6.2 is stable.

Since the measured time-response for $k_2 = 0.02$ V/rad is unstable and the measured time-response for $k_2 = 0.03$ V/rad is stable, it is concluded that the minimum stabilizing value of $k_2$ is given by $k_2 \in (0.02, 0.03]$ V/rad for $\tau_h = 0.32$, $k_1 = 15$ V/rad and $h = 2$ ms. This shows that the model-based result in Chapter 5 where the value of $k_2 = 0.028$ V/rad, is a good prediction of $k_2$ in practice.

The minimum stabilizing value of $k_2$ is determined for several constant delays $\tau_h$, a controller-gain $k_1 = 15$ V/rad and a sampling-time $h = 2$ ms. The results are depicted in Figure 6.3 which also shows the model-based stability region as determined in Chapter 5. Because the measured minimum stabilizing value of $k_2$ matches with the model-based results, it is concluded that the NCS model used in Chapter 5 is a good representation of the controlled PATO set-up with constant delays.

### 6.1.2 The maximum stabilizing value of $k_2$

The maximum stabilizing value of $k_2$ is determined by increasing the value of $k_2$ for several constant delays, a controller-gain $k_1 = 15$ V/rad and a sampling-time $h = 2$ ms, until the transition between a stable time-response and an unstable time-response is found. An example of a measured time-response is presented in Figure 6.4 where the measured time-response for controller-gains $k_1 = 15$ V/rad and $k_2 = 0.2$ V/rad, a constant delay $\tau_h = 0.32$ and a sampling-time $h = 2$ ms is given. This shows convergence of the measured angular position error to a neighborhood of zero. Similar to Figure 6.2, there is no exact convergence to zero. The control-input only saturates for $t \in [1, 1.05]$ s. Because there is no saturation of the control-input after the time-response is settled and because the angular position error converges to a neighborhood of zero, it is concluded that the measured time-response is stable. Compared to the time-response, as depicted in Figure 6.2, it is clear that less oscillations are needed to converge the angular position error in time. This is caused by an increased value of $k_2$, that increases the effective damping through velocity feedback. This decreases the settling-time.

Next, the value of $k_2$ is increased to $k_2 = 0.26$ V/rad and the measured time-response for controller-gains $k_1 = 15$ V/rad and $k_2 = 0.26$ V/rad, a constant delay $\tau_h = 0.32$ and a sampling-time $h = 2$ ms is depicted in Figure 6.4. This shows convergence of the measured angular position

![Figure 6.4](image_url)

**Figure 6.4**: The measured time-response for controller-gains $k_1 = 15$ V/rad and $k_2 = 0.2$ V/rad, a constant time-delay $\tau_h = 0.32$ and a sampling-time $h = 2$ ms.
error $e_{x,k}$ to a neighborhood of zero. In the steady state of the measured time-response, there is saturation of the control-input. Compared to Figure 6.3, the magnitude of the angular position error has increased. Due to the saturation of the control-input, it is not clear whether the convergence of the angular position error is caused by stability or because the saturated control-input limits the divergence of the angular position error in such a way that it seems that the time-response is stable. Therefore, an additional test is needed to determine if Figure 6.5 represents the time-response of a stable or unstable system.

If the time-response, as depicted in Figure 6.5 is stable, then the system is linear and it holds that $|E_x(j2\pi f)| = |S(j2\pi f)| \cdot |N(j2\pi f)|$, with $E_x(j2\pi f)$ the frequency spectrum of the measured angular position error $e_{x,k}$, $N(j2\pi f)$ the frequency spectrum of the measurement noise $n_k$ and $S(j2\pi f)$ the sensitivity function defined in (6.2) and (6.5), where the equivalent characteristics in the $z$-domain are given by the expression $z = e^{j2\pi f h}$, with $f$ the frequency and $h$ the sampling-time.

The frequency spectrum of the angular position error is derived with the Fourier transformation of the measured angular position error $e_{x,k}$. Because the motor rotates with a constant angular velocity, it is assumed that the measurement noise is only caused by the encoder resolution. Therefore, the frequency spectrum of the measurement noise $N(j2\pi f)$ is constant.

Next, the parameter combination $k_1 = 15$ V/rad and $\kappa = 0.32$ and a sampling-time $h = 2$ ms is chosen for which two situations are considered. First, the situation A with $k_2 = 0.2$ Vs/rad as considered in Figure 6.3. Recall that this represents a stable system. Second, the situation B with $k_2 = 0.26$ Vs/rad. Recall that situation B represents a system for which the stability could not be determined based on the time-response measurement (see Figure 6.5). If the situation B is stable, then both the situations A and B are given by:

\[
|E_{x,A}(j2\pi f)| = |S_A(j2\pi f)| \cdot |N(j2\pi f)|, \\
|E_{x,B}(j2\pi f)| = |S_B(j2\pi f)| \cdot |N(j2\pi f)|, \\
\]

with $E_{x,A}(j2\pi f)$ and $E_{x,B}(j2\pi f)$ the frequency spectrum of the measured angular position error $e_{x,k}$ of the situation A and B, respectively, $S_A(j2\pi f)$ and $S_B(j2\pi f)$ the sensitivity function corresponding to the situation A and B, respectively and $N(j2\pi f)$ the frequency spectrum of the measurement noise $n_k$. Then, the frequency spectrum of the measured angular position error $E_{x,B}$ can be estimated with:

\[
|E_{x,B}(j2\pi f)| = |S_B(j2\pi f)| \cdot |S_A(j2\pi f)|^{-1} \cdot |E_{x,A}(j2\pi f)|.
\]
Here, the frequency spectrum $E_{x,A}(j2\pi f)$ is derived with the Fourier transformation of the measured angular position error of situation A (see Figure 6.5) and the sensitivity functions $S_A(j2\pi f)$ and $S_B(j2\pi f)$ are defined in (5.11), (2.29) for the situation A and B, respectively. To determine whether or not the situation B is stable, the estimated frequency spectrum of the angular position error $\hat{E}_{x,B}$ can be compared to the frequency spectrum $E_{x,B}$, which is derived with the Fourier transformation of the measured angular position error of situation B (see Figure 6.5). The absolute values of both spectra, i.e. the estimated and measured frequency spectrum of the angular position error of situation B, are depicted in Figure 6.6. This shows that the spectrum of the measured angular position error matches with the spectrum of the estimated angular position error. Since both spectra are comparable, it is concluded that the measured time-response in Figure 6.5 is stable.

Next, the value of $k_2$ is increased to $k_2 = 0.28$ Vs/rad. The corresponding measured time-response is depicted in Figure 6.7 for a controller-gain $k_1 = 15$ V/rad, a constant delay $\frac{h}{2} = 0.32$ and a sampling-time $h = 2$ ms. This figure shows convergence of the measured angular position error to a neighborhood of zero. Compared to Figure 6.5, the magnitude of the angular position error and the amount of time that $u$ is saturated, after the time-response is settled, have increased. Because it is not clear whether the convergence of the angular position error is caused by stability or by the saturated control-input, relation (5.11) is used, where situation B refers to the case with $k_2 = 0.28$ Vs/rad and situation A to the case with $k_2 = 0.2$ Vs/rad, as before. The spectrum of the estimated angular position error and the spectrum of the measured angular position error are compared in Figure 6.7. This shows that, for frequencies $f \in (0, 95)$ Hz, the spectrum of the measured angular position error is larger than the spectrum of the estimated angular position error, i.e. $|E_{x}(j2\pi f)| > |\hat{E}_{x}(j2\pi f)|$. This means that the increase of the angular position error is not caused by an increase of the sensitivity to disturbances, but that it is caused by instability of the system. Therefore, it is concluded that the measured time-response of Figure 6.7 is unstable.

Since the measured time-response for $k_2 = 0.26$ V/rad is stable and the measured time-response for $k_2 = 0.28$ Vs/rad is unstable, it is concluded that the maximum stabilizing value of $k_2$ is given by $k_2 \in [0.26, 0.28]$ Vs/rad for $h = 0.32$, $k_1 = 15$ V/rad and $h = 2$ ms. This shows that the model-based result in Chapter 3 where the value of $k_2 = 0.269$ Vs/rad, is a good prediction of $k_2$ in practice.

The triangles in Figure 6.9 depict the minimum and maximum stabilizing values of $k_2$ for several constant delays $\frac{h}{2}$, a controller-gain $k_1 = 15$ V/rad and a sampling-time $h = 2$ ms. This figure also shows the model-based stability region as determined in Chapter 3. Because the measured maximum stabilizing value of $k_2$ matches with the model-based results, it is concluded that the

**Figure 6.7:** The measured time-response for controller-gains $k_1 = 15$ V/rad and $k_2 = 0.28$ Vs/rad, a constant time-delay $\frac{h}{2} = 0.32$ and a sampling-time $h = 2$ ms.

**Figure 6.8:** The spectra of the estimated and measured angular position error for controller-gains $k_1 = 15$ V/rad and $k_2 = 0.28$ Vs/rad, a constant time-delay $\frac{h}{2} = 0.32$ and a sampling-time $h = 2$ ms.
Figure 6.9: The measured minimum and maximum stabilizing values of $k_2$ for several constant delays $\tau_h$ and the estimated stability region in terms of $k_2$ and constant delay $\tau_h$ for a controller-gain $k_1 = 15$ V/rad and a sampling-time $h = 2$ ms.

model used in Chapter 3 is a good representation of the controlled PATO set-up with delays.

The validation of the stability analysis is performed for the controller-gain $k_1 = 15$ V/rad, because, for this value, the problem of identifying the maximum stabilizing value of $k_2$ is most clear and because the stability region is such that delays larger than the sampling-time can also be validated. The validation of the performance analysis is performed for the controller-gain $k_1 = 17$ V/rad. Therefore, the stability region is also validated for the value of $k_1 = 17$ V/rad. The result

Figure 6.10: The measured minimum and maximum stabilizing values of $k_2$ for several constant delays $\tau_h$ and the estimated stability region in terms of $k_2$ and constant delay $\tau_h$ for a controller-gain $k_1 = 17$ V/rad and a sampling-time $h = 2$ ms.
is presented in Figure 6.10, which shows the minimum and maximum stabilizing values of $k_2$ determined from the measurements for several constant delays $\tau_h$, a controller-gain $k_1 = 17 \text{ V/rad}$ and a sampling-time $h = 2 \text{ ms}$ and the model-based stability region as determined in Chapter 3. Similar to Figure 6.9, the measured minimum and maximum stabilizing values of $k_2$ match with the model-based results.

6.2 Settling-time

To obtain the settling-time of the measured time-response, an error-bound on the angular position error is considered. The angular position error will have an accuracy of the encoder resolution, which is $\pm \frac{\pi}{1000} \text{ rad}$, and that the error signal itself will also have an accuracy of the encoder resolution. Therefore, it is expected that the measured constant angular position error is within a bound of $\epsilon_x = \pm 2 \times 3.14 \cdot 10^{-3} \text{ rad}$. This is added with a margin of two times the encoder resolution and the desired error-bound is chosen $\epsilon_x = \pm 4 \times 3.14 \cdot 10^{-3} = \pm 0.01 \text{ rad}$.

First, the measured and simulated time-response are compared in Figure 6.11 for the controller-gains $k_1 = 17 \text{ V/rad}$ and $k_2 = 0.15 \text{ Vs/rad}$, a constant delay $\tau_h = 0.02$ and a sampling-time $h = 2 \text{ ms}$. Note that the tolerated angular position error is given by the gray area. The settling-time of the measured time-response is 56 ms. In Figure 6.11 it is noticed that the simulated time-response is faster than the measured time-response. This is caused by saturation of the control-input for the measured time-response, which limits the available power of the control system. Therefore, the system needs more time to respond on the angular position error. Another effect is that the measured maximum angular position error is larger than the simulated maximum angular position error. To study whether or not this effect is caused by the saturation of the control-input, a simulation that includes the saturation of the control-input is performed. This simulated time-response is compared to the measured time-response in Figure 6.12, which shows that there is only a relatively small difference between both time-responses. The difference is caused by the saturation of the control-input.

Figure 6.11: The measured and simulated time-response for controller-gains $k_1 = 17 \text{ V/rad}$ and $k_2 = 0.15 \text{ Vs/rad}$, a constant delay $\tau_h = 0.02$ and a sampling-time $h = 2 \text{ ms}$. In the simulated time-response, saturation of the control-input is not considered.

Figure 6.12: The measured and simulated time-response for controller-gains $k_1 = 17 \text{ V/rad}$ and $k_2 = 0.15 \text{ Vs/rad}$, a constant delay $\tau_h = 0.02$ and a sampling-time $h = 2 \text{ ms}$. The simulation accounts for the saturation of the control-input.
finite encoder resolution of the PATO set-up, which is not considered in the simulations. Here, it is concluded that the simulations, where the saturation of the control-input is taken into account, can be used to study the settling-time of the PATO set-up.

Next, the influence of the controller-gain $k_2$ on the settling-time is studied. Therefore, the settling-time is determined from the simulated and measured time-responses for several controller-gains $k_2$, a controller-gain $k_1 = 17$ V/rad, a constant delay $\tau = 0.02$ and a sampling-time $h = 2$ ms. This is depicted in Figure 6.13, which shows that for $k_2 \in [k_2, 0.15]$ Vs/rad the settling-time decreases and that for $k_2 \in [0.15, k_2]$ Vs/rad the settling-time increases, where $k_2$ and $k_2$ represent the minimum and maximum stabilizing value of $k_2$. This means that the smallest settling-time, for these values of $\tau, k_1$ and $h$, appears for $k_2 = 0.15$ Vs/rad. Here, it is clear that the measured settling-time matches with the simulated settling-times. Therefore, it is concluded that the simulation model, including the saturation of the control-input, represents the controlled PATO set-up with constant delays.

Since the influence of the controller-gain $k_2$ on the settling-time has now been validated, the influence of the constant delay $\tau$ on the settling-time is studied. Similar to Figure 6.13, the settling-time is determined from the simulated and measured time-responses for several constant delays $\tau$, the controller-gains $k_1 = 17$ V/rad and $k_2 = 0.15$ Vs/rad and a sampling-time $h = 2$ ms. The value of $k_2 = 0.15$ Vs/rad is chosen, because the smallest settling-time appears for this value. The results are depicted in Figure 6.14. This shows that the settling-time increases for an increasing constant delay $\tau$ and that the measured and simulated time-responses are comparable. In combination with the results on an increasing value of $k_2$, it is clear that the simulation model, with consideration of the saturated control-input, can be used to predict the settling-time of the closed-loop PATO set-up with delays.

### 6.3 Bandwidth

The bandwidth $f_b$ of the closed-loop PATO set-up is studied as has been done for the NCS model in Chapter 4. The open-loop frequency-response function $L(j2\pi f) = P(j2\pi f)K(j2\pi f)$ as defined in (2.29), (2.28) is derived using (6.1), (5.2) and the frequency-domain measurements. The bandwidth is determined for a range of controller-gains $k_2$ for several constant time-delays $\tau$, a controller-gain $k_1 = 17$ V/rad and a sampling-time $h = 2$ ms. The results are depicted in Figure 6.15 that also shows the corresponding bandwidth determined from the frequency-domain model (2.29). Note that, to ensure stability, for each delay $\tau$ only the range of $k_2 \in [k_2, k_2]$ is
used.

In Figure 6.15(a) the analytically obtained bandwidth is compared to the bandwidth obtained from the frequency-domain measurements for a constant time-delay $\tau_h = 0.02$. From this figure it is concluded that the bandwidth obtained from the measurements matches with the model-based result of Chapter 4. The same comparison is performed for constant delays $\tau_h = 0.32$ and $\tau_h = 0.62$ in Figure 6.15(b) and Figure 6.15(c) respectively. Similar to Figure 6.15(a) it is concluded that the analytically obtained bandwidth and the bandwidth obtained from the frequency-domain measurements match. This means that the frequency-response model of Chapter 4 can be used to predict the bandwidth of the closed-loop PATO set-up with delays.

### 6.4 Sensitivity

The sensitivity function of the PATO set-up is used to validate the robustness of the stability, because the sensitivity function is the reciprocal of the distance of the Nyquist curve from the critical point $-1$. To validate the robustness of the stability, the sensitivity function is determined with (6.1) from the frequency-domain measurements. From the sensitivity function, the sensitivity peak is derived using (3.2), because this is the reciprocal of the smallest distance to the critical point $-1$. Figure 6.16 depicts the analytically obtained sensitivity peak and the sensitivity peak obtained from the frequency-domain measurements for a range of controller-gains $k_2$ for several constant time-delays, a controller-gain $k_1 = 17$ V/rad and a sampling-time $h = 2$ ms. Note that, to ensure stability, for each delay $\tau$ only the range of $k_2 \in [k_2^-, k_2^+]$ is used.

Figure 6.16(a) shows the analytically obtained sensitivity peak and the sensitivity peak obtained from the frequency-domain measurements for a constant time-delay $\tau_h = 0.02$. Here, it is concluded that the analytically obtained sensitivity peak and the sensitivity peak obtained from the
frequency-domain measurements match. This means that, for these values, the robustness of the stability is validated. The same comparison is performed for constant time-delays $\tau_h = 0.32$ and $\tau_h = 0.62$, which are depicted in Figure 6.16(b) and Figure 6.16(c), respectively. Here, it is concluded that the frequency-response model of Chapter 3 can be used to predict the sensitivity of the PATO set-up.

### 6.5 Discussion

In this chapter, the time-domain and frequency-domain measurements of the PATO set-up are discussed and compared to the model-based results of Chapter 3 and Chapter 4. Note that the measurements are limited to the use of an output-feedback controller as defined in (2.22). First, it is shown that the stability region, which is obtained based on the eigenvalue analysis of the system matrix in of the controlled system (see (2.24)), is a useful prediction of the stability region in practice (in other words, the model-based and experimental stability regions match). Second, it is shown that the settling-time can be predicted if simulations of the system, including saturation of the control-input, are used. Without this saturation, the settling-time, obtained from simulations, is typically too low. Third, it is shown that the open-loop frequency-response model of (2.29) can be used to predict the bandwidth of the system in practice. Finally, it is shown that the stability robustness of the system can be predicted by studying the sensitivity as modeled in Chapter 3.

![Figure 6.16](image)

**Figure 6.16:** The analytically obtained maximum absolute value of the sensitivity function from (5.1) and the maximum absolute value of the sensitivity function obtained from the frequency-domain measurements for a range of controller-gains $k_2$, several constant delays $\frac{\tau}{h}$, a controller-gain $k_1 = 17$ V/rad and a sampling-time $h = 2$ ms.
Chapter 7

Stability analysis of NCSs with periodic time-delays

In this chapter, the influence of periodic time-delays on the stability of an NCS is investigated. In [1], it is shown that a bounded variation of the delays can result in an unstable system, although for each constant delay within these delay bounds the system is stable. This chapter presents similar experimental results for the output-feedback case to show that the variation in delays may indeed destabilize a system. To study NCSs with periodic delays, the NCS models with constant delays, as derived in Chapter 2, are expanded in Section 7.1 to incorporate periodic delays. In Section 7.2, the stability of the obtained NCS model is analyzed by studying the eigenvalues of the discrete-time PATO model in closed-loop and by studying simulated time-responses. The implementation of periodic delays on the PATO set-up is discussed in Section 7.3. In Section 7.4, the experimental results regarding closed-loop stability for time-varying delays is presented and compared to the analytic result obtained in Section 7.2. In Section 7.5, the obtained results on the stability analysis and validation are discussed.

7.1 The discrete-time NCS model

The periodic sequence of the delay for which a case is found that proves that a bounded variation of the delays can result in an unstable system, although for each constant delay within these delay bounds the system is stable, is given by \( \tau_a, \tau^b, \tau^b, \tau^b, \tau^b, \ldots \), with \( \tau^a \) and \( \tau^b \) constant and smaller than the sampling-time \( h \), i.e. \( 0 \leq \tau^a < \tau^b < h \). Note that the system, with fixed values of \( k_1, k_2 \) and \( h \), is stable for the constant delay \( \tau^a \) or \( \tau^b \). Recall that the discrete-time PATO model for constant time-delays is given by (2.2), with \( C = [1 \ 0] \) and \( x_k, \Phi, \Gamma_0(\tau) \) and \( \Gamma_1(\tau) \) defined in (2.16). This is used to determine the discrete-time PATO model with periodic delays:

\[
\begin{align*}
    x_{k+3} &= \Phi x_{k+2} + \Gamma_0(\tau^b)u_{k+2} + \Gamma_1(\tau^b)u_{k+1}, \\
    x_{k+2} &= \Phi x_{k+1} + \Gamma_0(\tau^b)u_{k+1} + \Gamma_1(\tau^b)u_k, \\
    x_{k+1} &= \Phi x_k + \Gamma_0(\tau^a)u_k + \Gamma_1(\tau^a)u_{k-1}, \\
    y_{k+2} &= Cx_{k+2},
\end{align*}
\]

with \( C = [1 \ 0] \) and \( x_k, \Phi, \Gamma_0(\tau) \) and \( \Gamma_1(\tau) \) defined in (2.16), with \( \tau \) replaced by either \( \tau^a \) or \( \tau^b \). Then, the discrete-time PATO model with periodic delays is given by:

\[
\begin{align*}
    x_{k+3} &= \Phi^3 x_k + \Gamma_0(\tau^b)u_{k+2} + \Gamma_1(\tau^b)u_{k+1} + \Gamma_2(\tau^b)u_k + \Phi^2 \Gamma_1(\tau^b)u_{k-1}, \\
    y_{k+2} &= C(\Phi^2 x_k + \Gamma_0(\tau^b)u_{k+1} + (\Phi \Gamma_0(\tau^a) + \Gamma_1(\tau^b))u_k + \Phi \Gamma_1(\tau^a)u_{k-1}),
\end{align*}
\]
with $\hat{\Gamma}_1 = \Phi\Gamma_0(\tau^b) + \Gamma_1(\tau^b)$ and $\hat{\Gamma}_2 = \Phi^2\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b)$.

The closed-loop, discrete-time PATO model for small delays is obtained by implementing the output-feedback controller (2.28) in the discrete-time model (7.1), which results in:

$$\xi_{k+3} = \Psi(\tau^b)\Psi(\tau^a)\xi_k,$$

(7.2)

with $\xi_k = [x_k \quad \varphi_{k-1} \quad \varphi_{k-2}]^T$ and $\Psi(\tau^a)$ and $\Psi(\tau^b)$ defined in (2.24), (2.25) with $\tau$ replaced by $\tau^a$ or $\tau^b$, respectively.

The models of (7.1) and (7.2) are applicable for time-domain analysis. Next, these models are used to obtain a frequency-domain model that is e.g. applicable to determine the sensitivity of the controlled system. Therefore, the open-loop frequency-response function of the PATO model is determined by deriving the transfer function between the angular position error $E_x(z)$ and the angular position output $Y_x(z)$. First, the transfer function between the control-input $U(z)$ and the angular position output $Y_x(z)$ of the discrete-time PATO model for periodic delays of (7.1) is defined as $Y(z) = \frac{1}{2}\Phi(\Gamma_0(\tau^b) + \Gamma_1(\tau^b))\Phi^2(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^3(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^4(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^5(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^6(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^7(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^8(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^9(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^{10}(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))U(z)$, with $\Phi$ replaced by either $\tau^a$ or $\tau^b$, this results in:

$$Y_x(z) = P(z)U(z),$$

(7.3)

where $P(z) = \frac{1}{2}\Phi(\Gamma_0(\tau^b) + \Gamma_1(\tau^b))\Phi^2(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^3(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^4(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^5(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^6(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^7(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^8(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^9(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))\Phi^{10}(\Gamma_0(\tau^a) + \Phi\Gamma_1(\tau^b))U(z)$, and $\Phi$ replaced by either $\tau^a$ or $\tau^b$. Next, the transfer function of the output-feedback controller given by (2.28) is combined with the transfer function given by (7.3). This results in the transfer function between the angular position error $E_x(z)$ and the angular position output $Y_x(z)$, which is given by:

$$Y_x(z) = L(z)E_x(z),$$

(7.4)

where $L(z) = P(z)K(z)$, with $P(z)$ defined in (7.3) and $K(z)$ defined in (2.25).

### 7.2 Stability analysis

To determine if a time-variation of the delays may result in instability, periodic delay sequences are investigated. For known sequences, the stability can be determined exactly because the stability analysis can be based on studying the systemmatrix eigenvalues of the closed-loop, discrete-time PATO model defined in (7.2), (2.24), (2.25). To prove the destabilizing effect, two constant delays are considered for which the system is stable. However, in combination according to a known sequence, these delays result in an unstable system. In the remaining part of this chapter, the system is evaluated for controller-gains $k_1 = 17$ V/rad and $k_2 = 0.23$ V/s, a sampling-time $h = 2$ ms and the periodic sequence given by $\tau^a$, $\tau^b$, $\tau^a$, $\tau^b$, $\tau^a$, $\tau^b$, ..., with the constant time-delays $\frac{\tau^a}{2} = 0.02$ and $\frac{\tau^b}{2} = 0.42$. To evaluate the stability for constant time-delays, the closed-loop model (2.24), (2.25), with $C = [1 \quad 0]$ and $\Phi$, $\Gamma_0$ and $\Gamma_1$ defined in (2.10), with either $\tau^a$ or $\tau^b$, is used. To evaluate the stability for periodic time-delays, the model of (7.2), (2.24), (2.25) is used.

First, the stability of the PATO set-up with a constant time-delay $\tau = \tau^a$ is determined. Therefore, the simulated time-response is depicted in Figure 7.4. This shows a stable behavior because the
angular position error $e_x$ converges to zero. The corresponding eigenvalues are $\lambda_{1,2} = 0.2982 \pm 0.7198i$, $\lambda_3 = -0.0004$ and $\lambda_4 = 0.8541$, which are all inside the unit-circle and therefore the closed-loop system is stable.

Next, the stability of the PATO set-up with a constant time-delay $\tau = \tau^b$ is evaluated. The simulated time-response is given in Figure 7.2, which shows stable behavior. The corresponding, stable, eigenvalues are $\lambda_{1,2} = 0.5284 \pm 0.8372i$, $\lambda_3 = -0.1049$ and $\lambda_4 = 0.8555$ (confirming stability).

The periodic delay sequence $\tau^a, \tau^b, \tau^b$ results in an unstable system, as is depicted in Figure 7.3. The instability can also be concluded from the eigenvalues of the system (7.2), (2.24), (2.25), which are $\lambda_1 = 0.6249$, $\lambda_2 = -1.0167$, $\lambda_3 = -0.5728$ and $\lambda_4 = 0$. Here, $\lambda_2$ is an unstable eigenvalue, because it is outside the unit-circle.

Because the PATO set-up is stable for constant delays $\tau^a$ and $\tau^b$ and unstable for a periodic delay $\tau^a, \tau^b, \tau^b, \tau^a, \tau^b, \tau^b, \tau^b, \tau^b, \ldots$, it is concluded that the time-variation of the delay may result in instability of a system.

Besides the studied periodic sequence of the delay, the destabilizing effect of other sequences can be investigated in a similar way, e.g. for $\tau^a, \tau^b, \tau^a, \tau^b, \ldots$ and $\tau^a, \tau^b, \tau^b, \tau^a, \tau^b, \tau^b, \tau^b, \tau^b, \tau^b, \ldots$. The periodic delay sequence $\tau^a, \tau^b, \ldots$ has no destabilizing effect (for the system and control parameters as studied above). The periodic delay sequence $\tau^a, \tau^b, \tau^b, \tau^b, \ldots$, however, does have a destabilizing effect. Therefore, it is concluded that the destabilizing effect does not only occur for one periodic sequence, but that it is a general problem.

Figure 7.1: The simulated time-response for a constant delay $\tau^a = 0.02$, controller-gains $k_1 = 17$ V/rad and $k_2 = 0.23$ V/s/rad and a sampling-time $h = 2$ ms.

Figure 7.2: The simulated time-response for a constant delay $\tau^b = 0.42$, controller-gains $k_1 = 17$ V/rad and $k_2 = 0.23$ V/s/rad and a sampling-time $h = 2$ ms.

Figure 7.3: The simulated time-response for a periodic delay $\tau^k = 0.02, 0.42, 0.42, 0.02, 0.42, 0.42, \ldots$, a controller-gain $k_1 = 17$ V/rad, a controller-gain $k_2 = 0.23$ V/s/rad and a sampling-time $h = 2$ ms.
7.3 Implementation of periodic time-delays

To validate the destabilizing effect of periodic delays obtained in the previous section, measurements are performed. To perform time-domain measurements, the control scheme as depicted in Figure 5.5 is considered. This scheme is used to study constant time-delays. To implement a time-variation of the delay, the constant delay $\tau$ is replaced by the scheme as presented in Figure 7.4.

Note that, due to the zero-order-hold block in Figure 5.5, the control-input $u_k$ has an effective sampling frequency of 500 Hz. In Figure 7.4, the control-input $u_k$ is delayed by the constant delays $\tau_a$ and $\tau_b$, which are given by either $N_a\%$ or $N_b\%$ of the sampling-time $h$. The pulse generator in Figure 7.4 is defined such that:

$$d(t) = \begin{cases} 1, & \text{if} \quad t \in [khp, khp + h) \\ 0, & \text{if} \quad t \in [khp + h, khp + 3h) \end{cases},$$

with $k = \left\lfloor \frac{k}{p} \right\rfloor$, which is the smallest integer smaller than or equal to $k/p$. $p$ is the size of the periodic sequence of the delay, $kh$ the sampling-instant and $h$ the sampling-time. The switch is open for input $a$, if $d > 0$, otherwise $b$ is used. This results in a periodic sequence of the delay as given by $\tau_a, \tau_b, \tau_a, \tau_b, \tau_a, \tau_b, \ldots$. Note that a constant time-delay is implemented if $\tau_a = \tau_b$.

Since the implemented control scheme is expanded, the internal delay might be changed. Therefore, the internal delay is determined again and it is concluded that the internal delay has not changed compared to the implementation of constant time-delays. This is shown in Appendix D.

7.4 Experimental validation of the stability analysis

To validate the analytical example of Section 7.2, time-domain measurements are performed on the set-up for controller-gains $k_1 = 17$ V/rad and $k_2 = 0.23$ Vs/rad and a sampling-time $h = 2$ ms. The measured time-response for the constant delay $\tau_a = 0.02$ is depicted in Figure 7.5. This time-response is stable, because the angular position error $e_{x,k}$ and control-input $u_k$ converge to a neighborhood of zero. Similar to Chapter 6, convergence to zero does not occur because of the measurement noise caused by the encoder resolution of $\frac{\pi}{1000}$ rad. The time-response in Figure 7.5 is considered as the reference time-response to evaluate the stability for the measured time-responses with constant delay $\tau_b$ and the periodic delay sequence $\tau_a, \tau_b, \tau_a, \tau_b, \tau_a, \tau_b, \ldots$.

Figure 7.6 depicts the measured time-response for the constant delay $\tau_b = 0.42$. Because the angular position error $e_{x,k}$ and control-input $u_k$ converge to a neighborhood of zero, the time-response is recognized as being stable. In Chapter 6, it was stated that if the measured time-response for $\tau_b$ is stable, the corresponding frequency spectrum of this time-response can be estimated from the frequency spectrum of the measured time-response for $\tau_a$. The estimation is performed using (6.1), with $E_{x,A}(j2\pi f)$ the frequency domain angular position error for $\tau_a$, $S_A(j2\pi f)$ the modeled sensitivity function for the PATO set-up with a constant delay $\tau_a$ and $S_B(j2\pi f)$ the modeled sensitivity function for the PATO set-up with a constant delay $\tau_b$.
frequency spectrum $E_{x,A}(j2\pi f)$ is determined by performing a Fourier transformation of the measured time-response, given in Figure 7.5. The sensitivity functions $S_A(j2\pi f)$ and $S_B(j2\pi f)$ are defined in (3.1), where $L(z)$ is defined in (2.29), with either $\tau^a$ or $\tau^b$. In Figure 7.7 the estimated frequency spectrum of the angular position error $\hat{E}_{x,B}(j2\pi f)$, as stated above, is compared to the spectrum of the measured angular position error $E_{x,B}(j2\pi f)$, as determined by a Fourier transformation of the time-response given in Figure 7.6 are compared. This shows that the spectrum of the measured angular position error is comparable to the spectrum of the estimated angular position error. Therefore, it is concluded that the measured time-response for the constant delay $\tau^b$ is stable indeed.

The measured time-response for the periodic delay $\tau^a$, $\tau^b$, $\tau^b$, $\tau^a$, $\tau^b$, $\tau^b$, $\ldots$ is depicted in Figure 7.8. This shows divergence of the angular position error $e_{x,k}$ and control-input $u_k$ until the control-input is saturated, which points to instability. Because the control-input is saturated after the time-response is settled, there is no divergence of the measured angular position error and it seems that the time-response is stable. Due to the saturation of the control-input, it is not clear whether the convergence of the angular position error is caused by stability or because the saturated control-input limits the divergence of the angular position error in such a way that it seems that the time-response is stable. Therefore, it is concluded that the stability of the time-response

**Figure 7.5:** The measured time-response for controller-gains $k_1 = 17$ V/rad and $k_2 = 0.23$ Vs/rad, a constant delay $\frac{\tau^a}{L} = 0.02$ and a sampling-time $h = 2$ ms.

**Figure 7.6:** The measured time-response for controller-gains $k_1 = 17$ V/rad and $k_2 = 0.23$ Vs/rad, a constant delay $\frac{\tau^b}{L} = 0.42$ and a sampling-time $h = 2$ ms.

**Figure 7.7:** The spectra of the estimated and measured angular position error for controller-gains $k_1 = 17$ V/rad and $k_2 = 0.23$ Vs/rad, a constant delay $\frac{\tau^b}{L} = 0.42$ and a sampling-time $h = 2$ ms.
in Figure 7.8 cannot be determined by studying the measured time-response only. Again, (6.1) is used to estimate the angular position error for the periodic delay sequence \( \tau^a, \tau^b \), \( \tau^b \) from the measured time-response for \( \tau^a \). In (6.1), \( E_{x,A}(j2\pi f) \) is the frequency domain angular position error for \( \tau^a \), given as the Fourier transformation of the measured time-response in Figure 7.8. \( S_A(j2\pi f) \) is defined in (3.1), where \( L(z) \) is defined in (2.29) with \( \tau = \tau^a \) and \( S_B(j2\pi f) \) is defined in (3.1), where \( L(z) \) is defined in (2.4) for the periodic delay sequence \( \tau^a, \tau^b, \tau^b \). The estimated spectrum \( \hat{E}_{x,B}(j2\pi f) \) is compared to the spectrum of the measured angular position error \( E_{x,B}(j2\pi f) \), as determined with the Fourier transformation of the measured time-response in Figure 7.8. The result is depicted in Figure 7.9 which shows a very small absolute value of the estimated angular position error for \( f \in (160,170) \) Hz. This is caused by a periodic change of the control-input \( u_k \), which is induced by the periodic time-delay. This results in several actuation frequencies that behave like sampling frequencies. Details on explaining the cause for the small absolute value of the estimated angular position error in Figure 7.9 are given in Appendix E. This shows the same effect for a stable situation with periodic delays, as depicted in Figure 7.2. If this is compared to Figure 7.9 it is concluded that, in Figure 7.9 the measured angular position error is also larger than the estimated angular position error (i.e. \( |E_x(j2\pi f)| > |\hat{E}_x(j2\pi f)| \)) for the frequencies \( f \in (0,100) \) Hz. Therefore, it is concluded that the increase of the angular position error is not caused by an increase of the sensitivity to disturbances, but that it is caused by instability of the system. Therefore, it is concluded that the measured time-response for the periodic delay \( \tau^a, \tau^b, \tau^b \), \( \tau^b, \tau^a, \tau^b, \tau^b, \ldots \) is unstable.

Because the measured time-responses for constant delays \( \tau^a \) and \( \tau^b \) are stable and the measured time-response for periodic delay \( \tau^a, \tau^b, \tau^b \) is unstable, it is proven that the time-variation of the delay may result in instability of a system.

### 7.5 Discussion

In this chapter, based on the NCS models of Chapter 2, models are presented for stability analysis of NCSs with periodic delays. Based on the NCS models with constant and periodic delays, it is shown that a time-variation in the delays may lead to instability, while for the constant upper- and lower bound of the periodic delays the NCS is stable. This effect is validated on the PATO set-up for a periodic sequence consisting of two different delays, both smaller than the sampling-time. First, measurements are presented for the two constant delays. It is shown that for both
constant delays the system is indeed stable. Second, measurements with a periodic sequence are presented and it is shown that this results in an unstable system. Therefore, it is concluded that, if time-variations in the delays occur, it is not sufficient to study the worst-case constant delay only. It is important to take the time-variation of the delay into account, especially because networks in a control system induce time-varying delays.
7.5 Discussion
Chapter 8

Conclusions and recommendations

This chapter presents the conclusions and some recommendations for future research in Section 8.1 and Section 8.2, respectively.

8.1 Conclusions

In this thesis, the analysis and experimental validation of an NCS with constant and periodic delays has been explored. First, a discrete-time, linear, time-invariant NCS model is presented based on an exact discretization of the linear plant at the sampling instants. This model is limited to constant delays that are either smaller or larger than the sampling interval. Then, this model is extended such that the control law is included. Ultimately, two models are distinguished, one model that describes an NCS with a state-feedback controller and a model that describes the NCS with an output-feedback controller. Based on these models, the stability and the performance, in terms of the settling-time and the bandwidth, of an NCS are predicted.

These performance criteria, in combination with the stability results, need to be considered during the controller design. Based on a single inertia example, as used in this thesis, the effect of the constant delay on the stability and performance is investigated. It is shown that for the state-feedback controller small time-delays may have a positive effect on the number of stabilizing controllers and, for most of these controllers, the settling-time decreases. However, the bandwidth is not influenced by an increase of the constant delay. For large delays, an increase of the delay may have a destabilizing effect and may lead to an increase of the settling-time. Again, the bandwidth is not influenced by the delay. For all delays, it is shown that the largest bandwidth is not achievable if the smallest settling-time is demanded, which makes a trade-off between these performance criteria necessary. For the output-feedback case, the same trade-off between the settling-time and the bandwidth holds. Note that here an increase of the delays always results in an increase of the settling-time and again the bandwidth is not affected. Moreover, a decrease of the number of stabilizing controllers is found for increasing delays.

Summarizing, the different analysis tools as discussed in this thesis allow for a trade-off between the settling-time and the bandwidth if a controller is designed that has to stabilize a system with known constant delays in the control loop. Moreover, it is possible to determine the maximum allowable constant time-delay for an NCS with a given controller guaranteeing stability and a certain performance. Such a reasoning is useful during the design of NCSs, because it gives allowable delays that may occur in the communication network. This may be helpful to choose a suitable communication network.

To ensure that the proposed stability and performance analysis tools are indeed useful for practical
applications, the different analytical results are validated for the output-feedback case. It is shown that the stability analysis results match, therefore the obtained analytical stability region is a useful prediction of the stability bounds in practice. For the settling-time, the original model and experimental results show a large difference, due to the saturation of the control input during the measurements. If, during the simulations, the saturation is taken into account, the predicted settling-time is a useful prediction of the settling-time in the actual set-up. For the bandwidth, the analysis and experimental results match. Therefore, it is concluded that the provided tools for stability analysis and performance in terms of the bandwidth and the settling-time are suitable to predict the stability and performance of NCSs with constant time-delays.

Based on the NCS models for constant time-delays, models are derived to predict the stability of NCSs with periodic delays. These models are limited to delays that are smaller than the sampling interval. Based on the stability analysis for an NCS with constant and periodic delays, it is concluded that a time-variation in the delays may lead to instability, while for the upper- and lower bound of the periodic delays the NCS is stable. This is experimentally validated. Therefore, it is concluded that, if time-variations in the delays occur, it is not sufficient to study the worst-case constant delay only.

8.2 Recommendations

In this report, the stability and performance analysis of NCSs have been validated for constant and periodic time-delays. Based on the conclusions in this thesis, the following research topics are ideas for future work:

- In this thesis, it is shown that it is not sufficient to study the worst-case constant delay only if the delay is time-varying. According to \[8\] and \[9\], the time-delays in NCSs are in general time-varying and the variation is not known in advance. Therefore, to study the stability of NCSs, both theoretically and experimentally, it is useful to consider time-varying delays. As a starting point, the overview papers \[6\], \[11\] and \[13\] can be used.

- In this thesis, only the analysis of the output-feedback controlled PATO set-up is validated, because only a position measurement is available in the PATO set-up. Therefore, not all the states are available and an output-feedback controller is used. Theoretically, it is shown that the stability region for the state-feedback case has a different shape than the stability region for the output-feedback case. This is especially the case for small delays, where the stabilizing behavior of constant delays was found. Due to the different shape of the stability region for the state-feedback case and the corresponding stabilizing behavior for small, constant time-delays, it is advised to experimentally validate the stability of the PATO set-up with a state-feedback controller. To measure the velocity, a set-up with both an encoder and a tachometer has to be considered. Another idea is to develop an observer that estimates the velocity more accurately, so that the state-feedback case is approached.

- During this research, saturation of the control-input for measured time-responses near the stability bounds appeared. Therefore, the stability of the measured time-response was hard to validate near the maximum stabilizing velocity feedback-gain. To avoid this problem, it is recommended that the saturation bounds on the control-input are large enough to make sure that the measured time-response can diverge before the control-input saturates. It is also possible that another reference input increases the difference between a stable and an unstable time-response near the stability bounds. Note that if a reference input with a changing direction is used (i.e. if the reference input consists of both a negative and positive velocity), the effect of the Coulomb friction needs to be taken into account.

- In this report, several criteria are presented for the performance and the stability robustness of the PATO set-up. Here, all performance criteria are investigated separately. This takes
time and more effort to design a suitable controller. Therefore, it is advised to develop a tool that combines all performance criteria to design a suitable controller for NCSs.
Appendix A

The Jury test

In this appendix, the Jury stability test \([12]\), which is the discrete-time equivalent of the Routh-Hurwitz \([3]\) stability criterium for continuous-time systems, is explained and applied to the PATO set-up. The Jury test is explained in Section A.1 and applied to the closed-loop, discrete-time PATO model with a state- and an output-feedback controller in Section A.2 and Section A.3 respectively. The Jury test is performed for the case without delays, i.e. \(\tau = 0\).

A.1 The Jury test

A discrete-time system is stable if all the roots of its characteristic polynomial lie inside the unit-circle. To test this condition without calculating the roots, the Jury test can be used \([4]\). Let \(a(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n\) be the characteristic polynomial of a discrete-time system. To apply the Jury test \([12]\), first make sure that \(a_0\) is positive by multiplying \(a(\lambda)\) by \(-1\), if necessary.

Now form rows of the coefficients as follows:

\[
\begin{array}{cccccc}
  a_0 & a_1 & \cdots & a_{n-1} & a_n \\
  a_n & a_{n-1} & \cdots & a_1 & a_0 \\
  b_0 & b_1 & \cdots & b_{n-1} & 0 \\
  b_{n-1} & b_{n-2} & \cdots & b_0 \\
  c_0 & c_1 & \cdots & 0 \\
\end{array}
\]

where the first and second rows are the coefficients of the characteristic polynomial in forward and reversed order, respectively. The third row is obtained by multiplying the second row by \(\frac{a_n}{a_0}\) and by subtracting this from the first row. The last element in the third row is thus zero and \(b_0 = a_0 - \frac{a_n}{a_0}a_n\), \(b_1 = a_1 - \frac{a_n}{a_0}a_{n-1}\) and so on. The fourth row is the third row in reversed order. Then, the fifth row is obtained analogously to the third row, however by multiplying the fourth row by \(\frac{b_{n-1}}{b_0}\) and by subtracting this from the third row. This gives \(c_0 = b_0 - \frac{b_{n-1}}{b_0}b_{n-1}\), \(c_1 = b_1 - \frac{b_{n-1}}{b_0}b_{n-2}\) and so on. Such a procedure is repeated until the last row consists of only one element. The needed number of rows is given by \(2n + 1\), with \(n\) the dimension of the characteristic polynomial. Using the Jury test, the discrete-time system is stable if the first element of all odd rows is positive, i.e. \(a_0, b_0, c_0, \ldots > 0\). This will be applied to the PATO model with a state-feedback or an output-feedback controller for the case without delays.
A.2 The non-delayed NCS with a state-feedback controller

To apply the Jury test, the characteristic polynomial of the discrete-time PATO model with a state-feedback controller, as given by (2.4), (2.8), (2.10), (2.18) is derived:

$$|\lambda - \Psi| = \left[ \begin{array}{cc} \lambda - 1 & \frac{k_2h^2}{J^*} \\ \frac{k_2h^2}{J^*} & \lambda - 1 + \frac{k_2h}{J^*} \end{array} \right] = 0,$$

which gives: $\lambda^2 + \left( \frac{k_2h}{J^*} + \frac{k_1h^2}{2J^*} - 2 \right) \lambda + 1 - \frac{k_2h}{J^*} + \frac{k_1h^2}{2J^*} = 0$. Thus, $a_0 = 1$, $a_1 = \frac{k_2h}{J^*} + \frac{k_1h^2}{2J^*} = 2$ and $a_2 = 1 - \frac{k_2h}{J^*} + \frac{k_1h^2}{2J^*}$.

The Jury scheme is formed using the coefficients of the characteristic polynomial, which gives that for stability it should hold that $a_0 > 0$, $b_0 > 0$ and $c_0 > 0$, where:

$$b_0 = a_0 - \frac{a_2^2}{a_0} \quad \text{and} \quad c_0 = a_0 - \frac{a_2^2}{a_0} - \left( \frac{a_1 - \frac{a_2}{a_0}}{a_0} \right)^2.$$

Using, $a_0 = 1$, $a_1 = \frac{k_2h}{J^*} + \frac{k_1h^2}{2J^*} = 2$ and $a_2 = 1 - \frac{k_2h}{J^*} + \frac{k_1h^2}{2J^*}$, this results in:

$$a_0 = 1, \quad b_0 = 1 - \left( 1 - \frac{k_2h}{J^*} + \frac{k_1h^2}{2J^*} \right)^2 \quad \text{and} \quad c_0 = 1 - \left( 1 - \frac{k_2h}{J^*} + \frac{k_1h^2}{2J^*} \right)^2.$$

This shows that $a_0 > 0$. First, it is determined when $b_0 > 0$. Therefore $1 - \left( 1 - \frac{k_2h}{J^*} + \frac{k_1h^2}{2J^*} \right)^2 = 0$ is derived, which gives: $k_2^2 - \left( \frac{2J^*}{h} + k_1h \right) k_2 + k_1 J^* + \frac{1}{4} h^2 k_1^2 = 0$. Solving controller-gain $k_2$ gives: $k_2 = \frac{2J^*}{h} + k_1h$ or $k_2 = \frac{k_1h}{2}$.

To determine the minimum and maximum stabilizing value of $k_2$, the sign of $b_0$ is depicted in Figure A.1(a) for several controller-gains $k_2$ and with $k_1 > 0$, $h > 0$ and $J^* > 0$. Here + indicates that $b_0 > 0$ and − indicates that $b_0 < 0$. Second, it is determined when $c_0 > 0$, by deriving $c_0 = 0$. This results in: $4 h k_2^2 - (8 k_1 h^2 + 8 J^*) k_2^2 + (8 J k_1 h + k_1^2 h^3) k_2 - 2 J k_1^2 h^2 = 0$. Solving controller-gain $k_2$ gives: $k_2 = \frac{2J^*}{h}$, $k_2 = \frac{h k_1}{2}$ or $k_2 = \frac{2J^*}{k_1}$.

\[ \begin{array}{cccccccccccc} - & - & + & + & + & + & + & + & + & - & - \\ \frac{h k_1}{2} & \frac{2J^*}{h} + \frac{h k_2}{2} \\ (a) & \text{Stabilizing controller-gains } k_2, \text{ according to } b_0 > 0. \\
+ & + & + & + & + & + & + & + & + & - & - \\ \frac{h k_1}{2} & \frac{2J^*}{h} \\ (b) & \text{Stabilizing controller-gains } k_2, \text{ according to } c_0 > 0. \end{array} \]

Figure A.1: Stabilizing controller-gains $k_2$, according to the Jury test.
To determine the minimum and maximum stabilizing value for \( k_2 \), the sign of \( c_0 \) is depicted in Figure A.1(b) for several controller-gains \( k_2 \) and \( k_1 > 0, h > 0 \) and \( J^* > 0 \). Here + indicates that \( c_0 > 0 \) and - indicates that \( c_0 < 0 \). Comparing Figure A.1(a) and Figure A.1(b) shows that, to guarantee that \( a_0 > 0, b_0 > 0 \) and \( c_0 > 0 \), it should hold that: \( \frac{k_1 h}{2h^2} < k_2 < \frac{2h^2}{h^2 - 4J^*} \).

### A.3 The non-delayed NCS with an output-feedback controller

To apply the Jury test, the characteristic polynomial of the discrete-time NCS model with an output-feedback controller (2.24), (2.25) is derived:

To apply the Jury test, the characteristic polynomial of the discrete-time NCS model with an output-feedback controller (2.24), (2.25) is derived:

\[
|\lambda I - \Psi| = \begin{vmatrix}
\lambda - 1 + \frac{1}{2J^*} (k_1 h^2 + k_2 h) & -h & -\frac{1}{2J^*} (k_2 h) \\
\frac{1}{J^*} (k_1 h + k_2) & \lambda - 1 & -\frac{1}{J^*} k_2 \\
-1 & 0 & \lambda
\end{vmatrix} = 0,
\]

which gives: \( \lambda^3 + \left( \frac{k_1 h^2}{2J^*} + \frac{k_2 h}{J^*} - 2 \right) \lambda^2 + \left( k_1 h^2 + 1 \right) \lambda - \frac{k_2 h}{2J^*} = 0 \). Thus, \( a_0 = 1, a_1 = \frac{k_1 h^2}{2J^*} + \frac{k_2 h}{J^*} - 2, a_2 = \frac{k_1 h^2}{2J^*} + 1 \) and \( a_3 = -\frac{k_2 h}{2J^*} \).

The Jury scheme is formed using the coefficients of the characteristic polynomial, which gives that for stability it should hold that \( a_0 > 0, b_0 > 0 \) and \( c_0 > 0 \), where:

\[
b_0 = a_0 - \frac{a_2^2}{a_0}, \quad c_0 = a_0 - \frac{a_2^2}{a_0} - \frac{(a_2 - a_3 a_1)}{a_0 - \frac{a_2^2}{a_0}}
\]

and

\[
d_0 = a_0 - \frac{a_2^2}{a_0} - \frac{(a_2 - a_3 a_1)}{a_0 - \frac{a_2^2}{a_0}} = \frac{(a_1 - a_3 a_2)}{a_0 - \frac{a_2^2}{a_0}} \left( \frac{a_1 - a_3 a_2}{a_0 - \frac{a_2^2}{a_0}} \right)^2.
\]

Using, \( a_0 = 1, a_1 = \frac{k_1 h^2}{2J^*} + \frac{k_2 h}{J^*} - 2, a_2 = \frac{k_1 h^2}{2J^*} + 1 \) and \( a_3 = -\frac{k_2 h}{2J^*} \), this shows that \( a_0 > 0 \). First, it is determined when \( b_0 > 0 \). This is derived by: \( 1 - \left( \frac{k_1 h^2}{2J^*} \right)^2 = 0 \). Solving controller-gain \( k_2 \) gives: \( k_2 = -\frac{2J^*}{h} \) or \( k_2 = -\frac{2h^2}{h^2 - 4J^*} \).

To determine the minimum and maximum stabilizing value for \( k_2 \), the sign of \( b_0 \) is depicted in Figure A.2(a) for several controller-gains \( k_2 \) and \( k_1 > 0, h > 0 \) and \( J^* > 0 \). Here + indicates that \( b_0 > 0 \) and - indicates that \( b_0 \leq 0 \). Second, the values of \( k_2 \) are determined for which holds that \( c_0 > 0 \). Therefore, \( c_0 = 0 \) is derived. This gives:

\[
(2h^4 k_1 - 8h^2 J) k_2^2 + (-4Jk_1 h^3 + 32J^2 k_2^2 + h^2 k_2^2) k_2^2 + (4J^2 k_2^2 h^4 - 32J^3 - 8J^2 h^2 k_1) k_2
+ 16J^3 k_1 h + 4J^2 k_2^2 h^3 = 0.
\]

Solving controller-gain \( k_2 \) gives: \( k_2 = \frac{2J^* (k_1 h^2 + 4J^*)}{h(k_1 h^2 - 4J^*)}, k_2 = \frac{-k_1 h^2 + 4J^* + \sqrt{h^2 k_1^2 - 24k_1 h^2 J^* + 16J^2}}{4h} \) or \( k_2 = \frac{-k_1 h^2 + 4J^* - \sqrt{h^2 k_1^2 - 24k_1 h^2 J^* + 16J^2}}{4h} \).

To determine the minimum and maximum stabilizing value for \( k_2 \), the sign of \( c_0 \) is depicted in Figure A.2(b) for several controller-gains \( k_2 \) and \( k_1 > 0, h > 0 \) and \( J^* > 0 \). Here + indicates that
A.3 The non-delayed NCS with an output-feedback controller

\[-\frac{2J}{h} \quad \frac{2J}{h} \]

(a) Stabilizing controller-gains \( k_2 \), according to \( b_0 > 0 \).

\[-k_1h^2 + 4J^* - \sqrt{h^4k_2^2 - 24kJ^*J^* + 16J^*^2} \]

\[-k_1h^2 + 4J^* + \sqrt{h^4k_2^2 - 24kJ^*J^* + 16J^*^2} \]

\[-k_2h^2 + 4J^* - \sqrt{h^4k_2^2 - 24kJ^*J^* + 16J^*^2} \]

\[-k_2h^2 + 4J^* + \sqrt{h^4k_2^2 - 24kJ^*J^* + 16J^*^2} \]

(b) Stabilizing controller-gains \( k_2 \), according to \( c_0 > 0 \).

(c) Stabilizing controller-gains \( k_2 \), according to \( d_0 > 0 \).

**Figure A.2:** Sign of controller-gain \( k_2 \) for controller-gain \( k_1 > 0 \), sampling-time \( h > 0 \), total inertia \( J^* > 0 \), determined with the Jury test.

\[ c_0 > 0 \text{ and } - \text{ indicates that } c_0 \leq 0. \text{ Third, it is determined when } d_0 > 0 \text{. Therefore } d_0 = 0 \text{ is derived, which results in:} \]

\[ k_2^2 + \frac{k_1h^2 - 4J}{2h} k_2 + Jk_1 = 0. \]

Solving controller-gain \( k_2 \) gives: \( k_2 = \frac{4J^* - k_1h^2 + \sqrt{16(J^*)^2 - 24kJ^*J^* + h^4k_2^2}}{4h} \) or \( k_2 = \frac{4J^* - k_1h^2 - \sqrt{16(J^*)^2 - 24kJ^*J^* + h^4k_2^2}}{4h} \).

To determine the minimum and maximum stabilizing value for \( k_2 \), the sign of \( d_0 \) is depicted in Fig-ure A.2(c) for several controller-gains \( k_2 \) and \( k_1, h, J^* > 0 \). Here + indicates that \( d_0 > 0 \) and − indicates that \( d_0 \leq 0 \). Comparing Figure A.2(a), Figure A.2(b) and Figure A.2(c) show that, to guaran-tee that \( d_0 > 0 \), \( b_0 > 0 \), \( c_0 > 0 \) and \( d_0 > 0 \), it should hold that: \( \frac{4J^* - k_1h^2 - \sqrt{16(J^*)^2 - 24kJ^*J^* + h^4k_2^2}}{4h} < k_2 < \frac{4J^* - k_1h^2 + \sqrt{16(J^*)^2 - 24kJ^*J^* + h^4k_2^2}}{4h} \).
Appendix B

Damping in the PATO set-up

In this appendix, the effect of damping in the bearings of the PATO set-up is studied. To determine the damping of this system, the time-responses to a reference signal of constant angular velocity is measured. First, it is studied if the damping is a Coulomb friction. Coulomb friction, or dry friction, is caused by the force, or resistance, that opposes the relative motion, or tendency toward such motion, of two surfaces in contact such as the bearings of the PATO set-up. To verify the appearance of the Coulomb friction of the PATO set-up, the dependence of the arithmetic mean of the measured angular position error, denoted by the mean angular position error, on the constant angular velocity of the reference signal to be tracked is studied. Therefore, the measured time-responses for several values of the constant angular velocity of the reference signal to be tracked,

\[ r_v = 75 \text{ rad/s} \]

(a)

\[ r_v = 100 \text{ rad/s} \]

(b)

\[ r_v = 125 \text{ rad/s} \]

(c)

\[ r_v = 150 \text{ rad/s} \]

(d)

Figure B.1: The measured time-responses for several values of the constant angular velocity of the reference signal to be tracked for controller-gains \( k_1 = 17 \text{ V/rad} \) and \( k_2 = 0.2 \text{ Vs/rad} \), \( \tau = 0 \) and a sampling-time \( h = 2 \text{ ms} \).
given by \( r_v \), for controller-gains \( k_1 = 17 \) V/rad and \( k_2 = 0.2 \) V/s, and a sampling-time \( h = 2 \) ms are depicted in Figure C.1. Figure B.1(a) shows that the mean angular position error for the value of \( r_v = 75 \) rad/s is \( \tau_x = 7.8 \) mrad. The mean angular position error for the value of \( r_v = 100 \) rad/s is \( \tau_x = 8 \) mrad, see Figure B.1(b). Figure B.1(c) shows that the mean angular position error for the value of \( r_v = 125 \) rad/s is \( \tau_x = 8.1 \) mrad and Figure B.1(d) shows that the mean angular position error for the value of \( r_v = 150 \) rad/s is \( \tau_x = 8.3 \) mrad. From Figure C.1 it is concluded that the mean angular position error is independent to the constant angular velocity of the reference signal to be tracked. Therefore, it is concluded that the damping in the bearings is a Coulomb friction.

To determine the Coulomb friction, the equation of motion in (2.15) is expanded and given by:

\[
J\ddot{x} = u - b\dot{x} \text{sign}(\dot{x}) - b\dot{r}_x \text{sign}(\dot{r}_x) + c_v \dot{x} \text{sign} (\dot{x}),
\]

(B.1)

with \( r_x \) the angular position of the reference signal to be tracked. For the control signal \( u \) it holds that:

\[
u = k_1 \dot{x} + k_2 \ddot{x},
\]

(B.2)

with \( k_1 \) and \( k_2 \) the controller-gains. Combining (C.1) and (C.2) gives \( J\ddot{x} + (b \text{sign}(\dot{x}) + c_v \dot{x} + c_v \dot{x} k_1) \dot{x} = J\ddot{r}_x + b\dot{r}_x \text{sign}(\dot{r}_x) \), when combined with the used reference signal, \( r_x = r_v, \dot{r}_x = \tau_x \), \( \dot{r}_x = 0 \), this results in: \( J\ddot{x} + (b \text{sign}(\dot{x}) + c_v \dot{x} + c_v \dot{x} k_1) \dot{x} + c_v \dot{x} k_1 \dot{x} = b\tau_x \text{sign}(\tau_x) \). The solution of this differential equation is:

\[
e_x(t) = A_0 e^{\lambda_1 t} + A_1 e^{\lambda_2 t} + \frac{b\tau_x}{k_1} \text{sign}(\tau_x),
\]

(B.3)

where \( A_0 e^{\lambda_1 t} + A_1 e^{\lambda_2 t} \) represents the impulse response and \( \frac{b\tau_x}{k_1} \text{sign}(\tau_x) \) represents the final value \[8\]. This final value corresponds to the measured mean angular position error in Figure C.1 i.e. \( \tau_x = \frac{b\tau_x}{k_1} \text{sign}(\tau_x) \). This is used to derive the Coulomb friction \( b \) from the measured time-responses as depicted in Figure C.1 with \( k_1 = 17 \) V/rad and the corresponding value of \( \tau_x \). The results are given in Table C.1.

<table>
<thead>
<tr>
<th>Angular velocity ( r_v ) [rad/s]</th>
<th>Mean angular position error ( \tau_x ) [mrad]</th>
<th>Damping coefficient ( b ) [V/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>7.8</td>
<td>1.77 · 10⁻³</td>
</tr>
<tr>
<td>100</td>
<td>8.0</td>
<td>1.36 · 10⁻³</td>
</tr>
<tr>
<td>125</td>
<td>8.1</td>
<td>1.10 · 10⁻³</td>
</tr>
<tr>
<td>150</td>
<td>8.3</td>
<td>0.94 · 10⁻³</td>
</tr>
</tbody>
</table>

Table B.1: The damping coefficients for several values of the constant angular velocity of the reference signal to be tracked.
Appendix C

Feedforward and the error-bound

In this appendix the feedforward and the error-bound are derived. The feedforward is used to compensate the viscous damping of the PATO set-up and the error-bound is the tolerance on the angular position error $e_{x,k}$. The feedforward and the error-bound are determined by studying the constant angular position error that appears if the reference signal is given as a constant angular velocity. Therefore, the equation of motion of the PATO set-up is rewritten, based on the error signal $e_x := r_x - \varphi$, as:

$$u = -J\dddot{e}_x - b\dot{e}_x + J\ddot{r}_x + b\dot{r}_x,$$  \hspace{1cm} (C.1)

with $r_x$ the angular position of the reference signal. For the control signal $u$ it holds that:

$$u = -k_1 e_x - k_2 \dot{e}_x,$$  \hspace{1cm} (C.2)

with $k_1$ and $k_2$ the controller-gains. Combining (C.1) and (C.2) gives $J\dddot{e}_x + (b - k_2)\dot{e}_x - k_1 e_x = J\ddot{r}_x + b\dot{r}_x$, when combined with the used reference signal, $r_x = r_v, \dot{r}_x = r_v, \ddot{r}_x = 0$, this results in: $J\dddot{e}_x + (b - k_2)\dot{e}_x - k_1 e_x = br_v$. The solution of this differential equation is:

$$e_x(t) = A_0 e^{\lambda_1 t} + A_1 e^{\lambda_2 t} + \frac{br_v}{k_1},$$  \hspace{1cm} (C.3)

where $A_0 e^{\lambda_1 t} + A_1 e^{\lambda_2 t}$ represents the impulse response and $\frac{br_v}{k_1}$ represents the final value. Since there is a constant error $e_x(t)$, the damping coefficient of the final value needs to be constant. To determine this, the damping coefficient is derived for several controller-gains $k_1$. This is done by measuring the constant part of the error, i.e. after the impulse response. Then the damping coefficient is derived with $\frac{br_v}{k_1}$. The results are given in Table C.1 and the corresponding measurements are given in Figure C.1.

<table>
<thead>
<tr>
<th>Controller-gain $k_1$ [V/rad]</th>
<th>Mean angular position error $\bar{e}_x$ [rad]</th>
<th>Damping coefficient $b$ [Vs/rad]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.135</td>
<td>$1.55 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>2</td>
<td>0.070</td>
<td>$1.40 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>3</td>
<td>0.045</td>
<td>$1.35 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>4</td>
<td>0.035</td>
<td>$1.40 \cdot 10^{-3}$</td>
</tr>
</tbody>
</table>
According to the measurements, the mean damping coefficient $\bar{b}$ is $1.375 \cdot 10^{-3}$ Vs/rad with a difference of $2.5 \cdot 10^{-5}$ Vs/rad. This difference might be caused by the encoder resolution of $\frac{2\pi}{2000}$ rad and is given by $b = \frac{k_1 \epsilon_{\text{encoder}}}{r_v}$, with $r_v = 100$ rad/s, $k_1 = 1$ V/rad and $\epsilon_{\text{encoder}} = \frac{\pi}{1000}$, this gives an accuracy of $3.14 \cdot 10^{-5}$ Vs/rad. Compared to the obtained difference of $2.5 \cdot 10^{-5}$ Vs/rad it is concluded that the difference is caused by the encoder resolution.

The feedforward is used to compensate the constant angular position error and is therefore given by $1.35 \cdot 10^{-3} \times 100 = 0.135$ V. This error will have an accuracy of the encoder resolution for several measurements and the error signal itself will also have an accuracy of the encoder resolution. So it is concluded that the measured constant angular position error is within a bound of $\epsilon_x = \pm 2 \times 3.14 \cdot 10^{-3}$ rad for several measurements. A margin is added of two times the encoder resolution, so the desired error-bound is chosen $\epsilon_x = \pm 4 \times 3.14 \cdot 10^{-3} = \pm 0.01$ rad.
Appendix D

The internal delay for the PATO set-up with periodic delays

Similar to Chapter 5, the internal delay is determined for the PATO set-up by performing frequency-domain measurements as represented in Figure 5.6, where the constant delay is replaced by the periodic delay as represented in Figure D.1.

The open-loop frequency-response of the PATO set-up for controller-gains $k_1 = 17 \text{ V/rad}$ and $k_2 = 0.2 \text{ Vs/rad}$, no time-delay, i.e. $\tau_1 = \tau_2 = 0$, and a sampling-time $h = 2 \text{ ms}$ is depicted in Figure D.1. Note that this corresponds to the constant delay case. Figure D.1 also shows the open-loop frequency-responses of several constant delays, given by $e^{-j(\frac{\tau}{2} + \tau_i)}$ [2], with $h$ the sampling-time and $\tau_i$ the internal delay. Here, it is concluded that the most comparable internal delay is $\frac{\tau}{2} = 0.02$, which corresponds to the constant delay case. To determine whether or not the internal delay is constant, the open-loop response of the PATO set-up is measured several times.

![Figure D.1](image)

**Figure D.1:** The measured open-loop frequency response of the PATO set-up compared to several constant delays $\frac{\tau}{2}$ for controller-gains $k_1 = 17 \text{ V/rad}$ and $k_2 = 0.2 \text{ Vs/rad}$, no time-delay, i.e. $\tau_1 = \tau_2 = 0$, and a sampling-time $h = 2 \text{ ms}$. 
It is concluded that the implementation of periodic delays does not influence the internal delay.
Appendix E

Spectrum periodic delays

In Figure 7.9 the frequency spectrum of the estimated angular position error is compared to the measured angular position error for an unstable situation. This shows that the absolute value of the estimated angular position error for frequencies $f \in [160, 170]$ is very small. This effect is studied in this appendix. To exclude the effect of an unstable system, a stable measured time-response is considered. Therefore, the time-response for the periodic delay sequence $\tau = \tau^a, \tau^b, \tau^b, \ldots$ is depicted in Figure E.1 for the controller-gains $k_1 = 17 \, \text{V/rad}$ and $k_2 = 0.23 \, \text{Vs/rad}$, the delays $\tau^a = 0.02$ and $\tau^b = 0.32$ and the sampling-time $h = 2 \, \text{ms}$. Because the angular position error $e_{x,k}$ and control-input $u_k$ converge to a neighborhood of zero, the time-response is recognized to being stable. Figure E.2 shows the corresponding spectra of the estimated angular position error and the measured angular position error.

Similar to Figure 7.9, a very small absolute value of the estimated angular position error appears for $f \in [160, 175] \, \text{Hz}$. To explain this extremely small value of $|E_x(j2\pi f)|$, the time instances at which a new control input is implemented on the system are compared. This is depicted in Figure E.3 which shows the control-input $u$ in time. In one periodic sequence, the control-input changes at the actuation times $t_k = kh + \tau^a$, $t_{k+1} = (k+1)h + \tau^b$, $t_{k+2} = (k+2)h + \tau^b$, and $t_{k+3} = (k+3)h + \tau^a$, with $t_{k+3}$ the beginning of a new periodic sequence. Instead of modeling
Figure E.3: The control-input for the periodic delay $\tau^a$, $\tau^b$, $\tau^b$, $\tau^a$, $\tau^b$, $\ldots$ in time.

the sampling instants, the actuation instants are considered. This lead to the following actuation intervals:

\[
\begin{align*}
    h_1 &= t_2 - t_1 = h + \tau^b - \tau^a = 2.8 \text{ ms}, \\
    h_2 &= t_3 - t_2 = h = 2.0 \text{ ms}, \\
    h_3 &= t_4 - t_3 = h - \tau^b + \tau^a = 1.2 \text{ ms}.
\end{align*}
\]

Each of these actuation intervals can be seen as a separate sampling frequency, which is denoted as the ‘implementation frequency’. Then, the implementation frequencies are $f_1 = 357 \text{ Hz}$, $f_2 = 500 \text{ Hz}$ and $f_3 = 833.3 \text{ Hz}$. Because frequency-response functions of discrete-time systems only hold for $f \leq \frac{1}{2T}$, the frequency-response function as depicted in Figure 7.9 holds for $f \leq \frac{357}{2} = 178.5 \text{ Hz}$. Therefore, it is concluded that the very small absolute value of the estimated angular position error for $f \in [160, 175] \text{ Hz}$ is a sampling effect caused by the implementation frequency $f_1 = 357 \text{ Hz}$. In practice, the frequency-response function is derived from measurements performed at the effective sampling frequency $f_s = 500 \text{ Hz}$. Therefore, the effect of the implementation frequency $f_1$ on the measured angular position error is very small.

As shown by Figure E.2, the effect of implementation frequencies induced by periodic delays does not only appear for an unstable system, but also for a stable system.
Bibliography


