Residual vibration suppression using Hankel Iterative Learning Control

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SUMMARY

In this paper, we present a new approach for suppression of residual vibrations in point-to-point motions based on lifted Iterative Learning Control (ILC). The approach is to add a signal to the command input during the point-to-point motion in order to compensate for residual vibrations. A special form of ILC with separate actuation and observation time windows is shown to converge to the required signal. Subsequently, we present ILC control strategies for residual vibration suppression in which convergence and performance specifications can be designed separately. Additionally, the designed controllers have the capability to constrain the amplitude of the command signal. The presented strategies are demonstrated on a flexible system and shown to be successful in suppression of residual vibration while minimizing the maximum amplitude of the command signal. Copyright \textcopyright 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In many applications, a flexible structure has to be positioned in order to perform an operation. The corresponding point-to-point motion can, however, introduce vibrations into the structure, thereby increasing settling time or degrading the reachable performance of the operation. In existing literature on input shaping, see [1, 2, 3] for overviews, the problem of suppressing residual vibrations is generally dealt with by convolving a designed input signal with a pulse sequence. This approach requires a relatively accurate model of the system if all vibration modes are to be suppressed. To somewhat relax this condition, adaptive techniques have been proposed, [4, 5, 6].

In this paper a different approach is suggested, which provides excellent suppression of residual vibrations, while requiring only limited a priori knowledge of the system. This
approach is based on Iterative Learning Control (ILC), e.g., [7, 8, 9, 10], which is a control strategy used to iteratively improve the performance of a repeated batch process. Improvement is achieved by updating the input signal from one experiment (trial) to the next, based on measurement data from earlier trials. Hence, instead of explicitly computing the input signal using a model, ILC converges to the sought input signal implicitly by executing a number of experiments, whereas the model in ILC is only required to provide convergence. Since the converged ILC input signal follows from experiments, it will in general outperform input signals which are explicitly based on a model.

Although ILC has mainly been designed to tackle servo problems, e.g., [11, 12, 13, 14], and in lesser extent to handle final output tracking problems (terminal ILC), [15, 16, 17], there are few results on ILC in combination with residual vibration suppression, [18, 19]. While the approach in [18, 19] is relatively complex, in this paper we show that residual vibration suppression in point-to-point motions can be easily achieved by introducing non-overlapping actuation and observation time windows in the ILC controlled system.

The main contributions of this paper are:

- Exploiting the ILC concepts for suppression of residual vibrations. Initial results have been reported in [20, 21, 22, 23].
- The introduction of non-overlapping actuation and observation time windows. The windows accommodate the formulation of the residual vibration problem and offer openings to several other applications.
- The paper shows that ILC design can be formulated such that performance and convergence demands can be satisfied separately, each specifying a different part of the control structure.
- In addition, the paper shows how asymptotic stabilizability can be guaranteed in the ILC design, thereby avoiding drift of marginally stable poles.

Although in this paper we use the results of the four issues to handle residual vibration suppression, the results of the latter two issues are more generally applicable in ILC.

The outline of this paper is as follows. In Section 2, the residual vibration suppression problem in point-to-point motions is defined. In Section 3, we initially present the ILC nomenclature, followed by the introduction of actuation and observation time windows which make ILC capable of properly handling point-to-point motion problems. Subsequently, we show that convergence and performance in ILC can be achieved separately in a control structure which is guaranteed asymptotically stabilizable. In Section 4, we present two ILC control strategies in which convergence and performance demands are combined with freedom to manipulate the input signal. We present results of the ILC control strategies applied to a flexible structure in Section 5, and finally, this paper ends with concluding remarks in Section 6.

2. PROBLEM FORMULATION

The problem of moving a flexible structure to a desired position and leaving it without residual vibration can be handled by a properly designed command signal. With the desire to be at
rest after completion of the motion, the command signal should accomplish suppression of this vibration during the point-to-point motion.

**Definition 1.** Point-to-point control problem:

*The design of a command signal actuating the system during the point-to-point motion resulting the system to be positioned at the desired position without residual vibrations after completion of the motion.*

In Figure 1, the point-to-point control structure is illustrated: for \( t \in [t(m_1), t(m_2)] \) the system is subjected to a command signal so that for \( t \in [t(n_1), t(n_2)] \) the system is at rest. This separation of actuation and observation is in correspondence with input shaping techniques.

![Figure 1. Reference signal with separate actuation and observation interval.](image)

Suppression of residual vibrations during the observation time interval can be considered as the compensation of disturbed initial conditions \( x(t_{n_1}) \) occurring at the beginning of the observation interval, i.e., at time \( t_{n_1} \). To see this, realize that actuation and observation take place in separate but adjacent time intervals. With actuation of the flexible structure limited to the actuation interval, the system behaves autonomously during the observation interval. Any nonzero response of this autonomous system (the residual vibrations) originates from disturbed initial conditions \( x(t_{n_1}) \) occurring at the initial time instant of the observation interval. Consequently, compensation for these disturbed initial conditions results in absence of any residual vibrations.

**Remark:** The focus of this paper is on residual vibration suppression in Single Input Single Output (SISO) systems. The presented theory is, however, easily extendable to Multi Input Multi Output (MIMO) systems, [24].

### 2.1. Point-to-point control problem

In this subsection, we formulate the problem of residual vibration suppression as compensation of a disturbed initial condition.

Given the discrete-time Linear Time Invariant (LTI) system \( J(t) \), \( t = 0, 1, 2, \ldots \) of minimal order

\[
J(t) : \begin{cases} 
x(t + 1) = Ax(t) + Bf(t) \\
y(t) = Cx(t) + Df(t),
\end{cases}
\]
with \( f(t) \) the command signal and \( y(t) \) the measured position. Then the convolutive mapping from \( f(t) \) to \( y(t) \) for \( t = [0, N - 1] \) is given by

\[
\begin{bmatrix}
y(0) \\
y(N-1)
\end{bmatrix} =
\begin{bmatrix}
D & 0 & \cdots & 0 \\
CB & D & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
CA^{N-2}B & \cdots & CB & D
\end{bmatrix}
\begin{bmatrix}
f(0) \\
f(N-1)
\end{bmatrix}.
\] (2)

Next, with the actuation interval defined by \( t \in [m_1, m_2] \), observation interval by \( t \in [n_1, n_2] \), and \( n_1 = m_2 + 1 \) (all in accordance with Figure 1), the convolutive mapping from \( f(t) \) during the actuation interval to \( y(t) \) during the observation interval equals

\[
\begin{bmatrix}
y(n_1) \\
y(n_2)
\end{bmatrix} =
\begin{bmatrix}
CA^{m-1}B & \cdots & CB \\
\vdots & \ddots & \vdots \\
CA^{n+m-2}B & \cdots & CA^{n-1}B
\end{bmatrix}
\begin{bmatrix}
f(m_1) \\
f(m_2)
\end{bmatrix},
\] (3)

\[ m = m_2 - m_1 + 1, \quad n = n_2 - n_1 + 1, \quad n_1 = m_2 + 1, \]

with \( J_H \in \mathbb{R}^{n \times m} \).

After reversing the input sequence in time, the system \( J_H \) corresponds to the Hankel operator which is known to have a rank equal to the order of the observable and controllable part of the underlying system, e.g., [25]. With this order equal to \( p \), for \( \min(m, n) > p \), the matrix \( J_H \) is thus rank deficient.

In case \( \text{rank}(J_H) = p < \min(m, n) \), we can represent \( J_H \) as the product of two full rank matrices using full rank decomposition,

\[
J_H = JLJR, \quad \text{with} \quad JL \in \mathbb{R}^{n \times p}, \quad JR \in \mathbb{R}^{p \times m},
\] (4)

representing the following two mappings

\[
\begin{bmatrix}
y(n_1) \\
y(n_2)
\end{bmatrix} = JLx_{n_1}, \quad x_{n_1} = JR\begin{bmatrix}
f(m_1) \\
f(m_2)
\end{bmatrix}.
\] (5)

with, e.g.,

\[
J_L = \begin{bmatrix}CT & (CA)^T & \cdots & (CA^{n-1})^T \end{bmatrix},
\]

and

\[
J_R = \begin{bmatrix}A^{m-1}B & A^{m-2}B & \cdots & B \end{bmatrix}.
\]

In (5), \( x_{n_1} \in \mathbb{R}^p \) is the state vector at \( t = n_1 \). Rewriting (5), we have

\[
y = JLx_{n_1}, \quad x_{n_1} = J_L^T y_L = (J_L^T J_L)^{-1}J_L^T y_L,
\] (6)

with \( J_L^T \) the Moore-Penrose inverse of \( J_L \). Taking, without loss of generality, the desired position during the observation interval equal to \( y_{ref} = 0 \), the system has zero residual vibrations if \( y(t) = y_{ref} \) for \( t \in [n_1, n_2] \). With \( y = 0 \), we have \( x_{n_1} = 0 \).

Previous reasoning shows that residual vibrations can be considered the result of disturbed initial condition \( x_{n_1} \). Consequently, suppression of the residual vibrations can be accomplished by compensating for \( x_{n_1} \).
3. RESIDUAL VIBRATION SUPPRESSION WITH ILC

In this section, we first present the ILC nomenclature. Subsequently, we introduce the actuation and observation window as tools to define an ILC control problem capable of suppressing residual vibrations. This results in the presentation of the general ILC control structure, which is guaranteed to be asymptotic stabilizable. Finally, we derive convergence and performance properties of this control structure.

3.1. Iterative Learning Control

The operations handled by ILC are assumed repetitive, and hence each trial covers a finite time span. In discrete-time lifted ILC a time signal \( y_k(t) \) for \( t = 0, 1, \ldots, N - 1 \) is given by

\[
y_k = \begin{bmatrix} y_k(0) & y_k(1) & \cdots & y_k(N - 1) \end{bmatrix}^T,
\]

with \( k \) the trial number and \( N \) the total number of time samples in the trial. The dynamic system of (1) in lifted notation corresponds to (2) and is given by

\[
y_k = Jf_k.
\]

Furthermore, the ILC control structure used in this paper is based on the widely used ILC algorithm, \([7]\),

\[
f_{k+1} = f_k + \Delta f_k = f_k + Le_k,
\]

with \( r \) the reference signal, \( e_k = r - y_k \) the error, and matrix \( L \) the learning controller. Note that the vector \( r \) is not a function of \( k \), indicating that the reference is constant in trial domain.

Combining (7) and (8) gives us the general ILC control problem of Figure 2. The matrices \( W_i \) and \( W_o \) define additional actuation and observation time windows which can be used to influence the behavior of the system (default \( W_i = W_o = I_N \)), vector \( u_k \) is the trial state with \( u_0 = 0 \), and \( w^{-1}I_N \) denotes a one trial delay. With an initial command signal \( f_{init} \) constant in trial domain, the output \( y_{init} = Jf_{init} \) is trial invariant, and hence can be seen as part of the exogenous signal \( r - Jf_{init} \).

![Figure 2. General Lifted ILC scheme including actuation window matrix \( W_i \) and observation window matrix \( W_o \).](image)

Including the window matrices, the trial domain dynamics are given by

\[
\begin{align*}
    u_{k+1} &= (I_N - LW_o JW_i)u_k + LW_o(r - Jf_{init}) \\
    f_k &= W_i u_k.
\end{align*}
\]
Remark: Though the ILC control algorithm is recursive in nature, we use the term “feedback control” to address ILC control in trial domain, Figure 2, and “asymptotic stability” to address asymptotic convergence of the recursive algorithm, conform ILC literature.

For ILC to function properly, asymptotic stability of the feedback system is essential, e.g., [26]. With (9) describing the evolution of the system in trial domain and \( u_k \) representing the trial state, asymptotic stability corresponds to \( \lim_{k \to \infty} u_k = 0 \) \( \forall u_0 \) and \( r - y_{init} = 0 \). A standard result of stability theory now states that asymptotic stability is achieved when the coefficient matrix of (9) has spectral radius smaller than one, i.e. \( \rho(I_N - K) < 1 \) for \( K = LW_oJW_i \), see Figure 3.

![Figure 3. General feedback scheme.](image)

3.2. Hankel Iterative Learning Control

To make ILC capable of handling disturbed initial conditions, the system of (7) is first transformed to \( J_H \) of (3). This is accomplished by defining \( W_i \) and \( W_o \) to be

\[
W_i = \begin{bmatrix} 0 & I_m \end{bmatrix}^T, \quad W_o = \begin{bmatrix} 0 & 0 \end{bmatrix},
\]

resulting in \( W_oJW_i = J_H \). The exogenous signal \( \tilde{r} := W_o(r - Jf_{init}) \) now corresponds to the residual vibrations during the observation interval. Due to the properties of system \( J_H \), we will denote ILC applied to point-to-point motion problems as Hankel ILC.

A novel, general control structure for Hankel ILC is presented in Figure 4, with the ILC controller consisting of two parts: \( L_L \in \mathbb{R}^{n \times p} \) and \( L_R \in \mathbb{R}^{p \times m} \). The corresponding open and closed loop trial domain dynamics are given by (12) and (13), respectively.

\[
u_{k+1} = u_k + L_L(\tilde{r} - y_k) \quad (12)
\]

\[
u_{k+1} = (I_p - L_LJ_HL_R)u_k + L_L\tilde{r}. \quad (13)
\]

The structures (9), Figure 2, and (13), Figure 4, are equivalent in the sense that the loop gains obtained by opening the loop in between \( J_L \) and \( J_R \) inside \( J_H = J_LJ_R \) can be made identical. These loop gains are

\[
J_R(wI_N - I_N)^{-1}LJ_L, \quad \text{and} \quad J_RL_R(wI_p - I_p)^{-1}L_LJ_L.
\]
respectively, and are equal if

$$J_R L J_L = J_R L_R L_L J_L$$  \hspace{1cm} (14)$$

In the above reasoning, we maintain the assumption $p = \text{rank}(J_H)$. Then given $L$, the choices $L_R = J_R^T (J_R J_R^T)^{-1}$, $L_L = J_R L$ verify (14). Conversely, given $L_L$ and $L_R$, the choice $L = L_R L_L$ verifies (14). The step (9) \rightarrow (13) has removed integrators in the loop that are not controllable/observable. This renders (13) asymptotically stabilizable.

**Lemma 1.** There exist constant $L_L$ and $L_R$ such that the system (13), Figure 4, is asymptotically stable in trial domain, if and only if $p \leq \text{rank}(J_H)$.

The proof follows directly by realizing that Lemma 1 states that, given the control structure of Figure 4, maximally rank$(J_H)$ integrators can be stabilized in trial domain.

Since asymptotic stability of the control structure of Figure 2 can not be guaranteed due to Lemma 1, we will focus on Hankel ILC with the control structure of Figure 4.

### 3.3. Asymptotic stability and performance

In this subsection, we focus on the properties of asymptotic stability and performance of the general control structure of Figure 4. For Lemma 2 and Lemma 3 we choose $p = \text{rank}(J_H)$, thereby satisfying Lemma 1.

**Lemma 2.** Given $L_L$, then arbitrary pole placement in (13) by choice of $L_R$ is possible, if and only if $\text{rank}(L_L J_L) = p$. Conversely, given $L_R$, then arbitrary pole placement in (13) by choice of $L_L$ is possible, if and only if $\text{rank}(J_R L_R) = p$.

**Proof**

Recall the closed loop trial dynamics (13)

$$u_{k+1} = (I_p - (L_L J_L)(J_R L_R)) u_k + L_L \tilde{f},$$

with $L_L J_L \in \mathbb{R}^{p \times p}$ and $J_R L_R \in \mathbb{R}^{p \times p}$. Given $L_L$, pole placement in system (13) is possible by $L_R$, if and only if $(I_p, L_L J_L)$ is controllable, $(I_p, L_L J_L)$ controllable $\iff$ rank$(L_L J_L) = p$.

Similarly, given $L_R$, pole placement in system (13) is possible by $L_L$, if and only if $(J_R L_R, I_p)$ is observable, $(J_R L_R, I_p)$ observable $\iff$ rank$(J_R L_R) = p$. \qed
For the ILC controlled system in Figure 4, the expressions for performance $J$ and optimal performance $J_{opt}$ are given by (15) and (16), respectively.

$$J(L_L, L_R) = \lim_{k \to \infty} \|e_k\|_i, \quad (15)$$

$$J_{opt} = \min_{L_L, L_R} J(L_L, L_R), \quad (16)$$

with $\| \cdot \|_i$ any compatible norm.

**Lemma 3.** Assume system (13) is asymptotically stable, then performance is a function of $L_L$ only. More specifically, optimal performance is achieved with $L_L = J_{L}^{\dagger} = (J_L J_L)^{-1} J_L$.

**Proof**

Due to stability, $u_\infty = \lim_{k \to \infty} u_k$ exists and satisfies

$$u_\infty = (I_p - L_L J_H L_R) u_\infty + L_L \tilde{r}$$

$$u_\infty = (L_L J_H L_R)^{-1} L_L \tilde{r}. \quad (17)$$

Consequently, with (17), we can express $e_\infty = \lim_{k \to \infty} e_k$:

$$e_\infty = \tilde{r} - J_H L_R u_\infty$$

$$e_\infty = (I_n - J_L J_R L_R (L_L J_L J_R L_R)^{-1} L_L) \tilde{r}$$

$$e_\infty = (I_n - J_L J_R L_R (J_R L_R)^{-1} (L_L J_L)^{-1} L_L) \tilde{r}$$

$$e_\infty = (I_n - J_L (L_L J_L)^{-1} L_L) \tilde{r}. \quad (18)$$

This shows that performance only depends on $L_L$.

Take $\tilde{r}$ parameterized as

$$\tilde{r} = \tilde{r}_1 + \tilde{r}_2 = P \tilde{r} + (I_n - P) \tilde{r}, \quad (19)$$

with $P = J_L J_L^{\dagger}$ the projection matrix $P : \mathbb{R}^n \to \text{Im}(J_L)$, $\tilde{r}_1 \in \text{Im}(J_L)$ and $\tilde{r}_2 \in \text{Im}(J_L)^\perp$. As $y_k \in \text{Im}(J_L)$, optimal performance is obtained for $e_\infty \in \text{Im}(J_L)^\perp$, i.e., for $e_\infty = \tilde{r}_2 = (I_n - P) \tilde{r}$.

Propose $L_L = J_L^{\dagger}$, then (18) becomes

$$e_\infty = (I_n - J_L (J_L^{\dagger} J_L)^{-1} J_L^{\dagger}) \tilde{r} = (I_n - J_L J_L^{\dagger}) \tilde{r}. \quad (20)$$

With (20) equal to $\tilde{r}_2$, this completes the proof. □

While asymptotic stability can be imposed by either $L_L$ or $L_R$ (Lemma 2), performance is dictated by the choice of $L_L$ only (Lemma 3). Hence, in the ILC control design for the system presented in Figure 4, we are free to associate $L_L$ with performance and associate $L_R$ with asymptotic stability. As a result, for the ILC control problem of Figure 4, performance and asymptotic stability demands can be satisfied separately by the design of $L_L$ and $L_R$, respectively.

### 4. CONTROL DESIGN

Having discussed design considerations for $L_L$ in Lemma 3, in this section we propose two control strategies for $L_R$. Both strategies focus on asymptotic stability and on shaping of the command signal to minimize its maximum amplitude.
Based on Lemma 3, \( L_L \) is designed to achieve optimal performance:

\[
L_L = J_L^T = (J_L^T J_L)^{-1} J_L^T,
\]

With \( L_L J_H = J_R \) and \( L_L \tilde{r} = J_L^T \tilde{r} \), the trial domain dynamics (13) can be expressed as

\[
u_{k+1} = (I_p - J_R L_R) u_k + J_L^T \tilde{r}.
\]

**Lemma 4.** \( L_R \) control design in Hankel ILC consists of two steps, one governing asymptotic stability, the other governing command shaping. Both parts can be designed separately without loss of generality.

**Proof**

Consider (22) and define \( K := J_R L_R \) with \( \rho(I_p - K) < 1 \). Then the general solution for \( J_R L_R = K \) is given by [27],

\[
L_R = J_R^{(1)} K + (I_m - J_R^{(1)} J_R) Y,
\]

with \( Y \in \mathbb{R}^{m \times p} \) arbitrary, and \( J_R^{(1)} \) any \( (1) \)-inverse of \( J_R \), i.e., satisfying \( J_R J_R^{(1)} J_R = J_R \). Note that \( \text{Im}(X_1) = \text{ker}(J_R) \perp \text{Im}(X_2) = \text{ker}(J_R) \).

With \( X_2 \) fully described by \( J_R \), \( L_R \) is fully defined by \( X_1 \) and \( Y \). With \( X_1 \cap X_2 Y = \emptyset \), \( X_1 \) and \( X_2 Y \) each define a separate part of \( L_R \). Therefore, design of \( X_1 \) or \( Y \) can be done separately.

Using (23), the trial domain dynamics are given by

\[
u_{k+1} = (I_p - J_R X_1) u_k + J_L^T \tilde{r}
\]

\[
f_k = X_1 u_k + X_2 Y u_k,
\]

since \( J_R X_2 = 0 \).

From (24) it is clear that asymptotic stability of Hankel ILC corresponds to \( \rho(I_p - J_R X_1) < 1 \). Furthermore, the command signal of (25) can be shaped by \( Y \). Since \( X_1 \) and \( Y \) can be designed separately, asymptotic stability and command shaping can be designed separately.

\[\square\]

**Remark:** For clarity, in this paper we design \( L_R \) based on \( L_L = J_L^T \). Any other choice for \( L_L \) requires replacement of \( X_1 \) with \( X_1 (L_L J_L)^{-1} \).

Based on Lemma 4, we first design an asymptotically stable Hankel ILC controller by properly designing \( X_1 \), and additionally shape the command signal using \( Y \).

4.1. Step \( X_1 \): asymptotic stability

As shown in Lemma 4, asymptotic stability of the system depends on \( \rho(I_p - J_R X_1) \). A straightforward solution for \( X_1 \) is given by inverse model-based control. When placing all trial domain poles at \( 1 - \gamma \), the matrix \( X_1 \) is given by

\[
I_p - J_R X_1 = (1 - \gamma) I_p \rightarrow X_1 = \gamma J_R^T,
\]

where \( 0 < \gamma < 2 \) is the so-called learning gain, and \( J_R^T = J_R^T (J_R J_R^T)^{-1} \) the Moore-Penrose inverse of \( J_R \). For \( \gamma = 1 \) we have deadbeat control, meaning that convergence to steady state behavior, i.e., the behavior \( u_k \) for \( k \rightarrow \infty \), is achieved after only one trial.
A different approach to designing $X_1$ can be found by applying LQ optimal control. Without showing the derivation, the LQ-optimal solution for $X_1$ can equal

$$X_1 = \gamma \left( J_R^T J_R + \beta I_m \right)^{-1} J_R^T, \quad \beta > 0$$

$$= \gamma J_R^T \left( J_R J_R^T + \beta I_p \right)^{-1} \beta \geq 0, \quad (27)$$

with $\beta$ and $\gamma$ design parameters, [12].

In case the system $J_R$ contains model uncertainties, we can use $\gamma$ and $\beta$ to make the feedback system robustly asymptotically stable in trial domain. By decreasing $\gamma$ (with default $\gamma = 1$) and/or increasing $\beta$ the feedback system converges at a lower rate (the trial domain poles are placed closer to the unit circle). Conversely, the feedback system becomes less sensitive to model uncertainties in $J_R$. Note, however, that the values for $\gamma$ and $\beta$ in $X_1$ of (27) do not influence the performance $e_\infty$ (Lemma 3).

4.2. Step Y: command shaping

In this paper, the goal of command shaping is to reduce the larger amplitudes in the command signal. By reducing the amplitude, we reduce the possibility of saturation of the command signal. We propose two approaches to reach this goal: The first one is based on minimization of the maximum amplitude of the command signal, referred to as $\ell_\infty$ control, [28]. The second approach is related to minimization of the weighted converged command signal $f_\infty$, denoted as $\ell_2$ control.

$\ell_\infty$ control

With $f_{\text{init},m} := W_i f_{\text{init}}$ the initial command signal during the actuation interval, the total command signal applied to the system during the actuation interval equals

$$f_{k,\text{tot}} = f_k + f_{\text{init},m} = X_1 u_k + X_2 Y u_k + f_{\text{init},m}. \quad (28)$$

The terms $f_{\text{init},m}$ and $X_1 u_k$ in (28) are known, while $X_2 Y u_k$ is still undefined. Since there are no restrictions imposed on $Y$, we can alter $X_2 Y u_k$ to $X_2 \theta_k$ making it independent of $u_k$. Now the goal of minimizing the maximum command amplitude can be expressed by $\min(J)$, with

$$J = \max |f_{k,\text{tot}}| = \max_\theta |X_1 u_k + f_{\text{init},m} + X_2 \theta|. \quad (29)$$

This minimax optimality problem can be rewritten as a linear programming (LP) problem

$$\min \alpha, \text{ subject to } -\alpha \leq F_k + X_2 \theta_k \leq \alpha, \quad (30)$$

with $F_k := X_1 u_k + f_{\text{init},m}$. A general notation for (30) is given by

$$\min g^T z_k, \text{ subject to } A z_k \leq b \quad (31)$$

with $g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} X_2 & -1 \\ -X_2 & -1 \end{bmatrix}$, $b = \begin{bmatrix} -F_k \\ F_k \end{bmatrix}$, and $z_k = [\theta_k \alpha]^T. \quad (32)$

The total command signal applied to the system during the actuation interval in trial $k + 1$ can be found by applying the optimal $\theta_{k+1,\text{opt}}$ in

$$f_{k+1,\text{tot}} = X_1 u_{k+1} + X_2 \theta_{k+1,\text{opt}} + f_{\text{init},m}. \quad (33)$$
with \( u_{k+1} \) from (12).

\( \ell_2 \) control
In \( \ell_2 \) control, an analytical solution for \( L_R \) is found. With \( X_1 \) defined, we focus on finding an analytical expression for \( Y \) in \( X_2 Y u_k \).

Using (17), the converged command signal is expressed by

\[
\begin{align*}
f_\infty &= L_R (J_R^T J_R L_R)^{-1} J_R^T \tilde{r}, \\
&= L_R (J_R X_1)^{-1} J_R^T \tilde{r}.
\end{align*}
\]

In \( \ell_2 \) control, we choose to focus on minimizing the weighted \( f_\infty \), (35): \( \min ||f_\infty||_W^2 \). The corresponding \( \ell_2 \) objective \( J \) is given by

\[
J = s^T (X_1^T + Y^T X_2^T) W (X_1 + X_2 Y) s
\]

With \( X_2^T W X_2 \) square and rank deficient we cannot directly determine an expression for \( Y \) from (38).

A solution for \( Y \) can be found by applying Singular Value Decomposition, defined by

\[
J_H = U \Sigma V^T = [U_1 \quad U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma_1 V_1^T,
\]

with \( J_H \in \mathbb{R}^{n \times m}, \ U_1 \in \mathbb{R}^{n \times p}, \ \Sigma_1 \in \mathbb{R}^{p \times p}, \ V_1 \in \mathbb{R}^{m \times p} \).

Using (39), we can define:

\[
J_L := U_1, \quad J_R := \Sigma_1 V_1^T, \quad J_R^{(1)} := V_1 \Sigma_1^{-1}
\]

\[
X_2 = I_m - J_R^{(1)} J_R = I_m - V_1 V_1^T = V_2 V_2^T.
\]

With \( X_2 \) from (40), \( Y \) can be expressed as

\[
Y = -V_2 (V_2^T W V_2)^{-1} V_2^T W X_1.
\]

Hence, \( L_R \) is given by

\[
L_R = (I - V_2 (V_2^T W V_2)^{-1} V_2^T W) X_1.
\]
To compare our findings with the point-to-point ILC controller given in [20], we present their LQ optimal ILC controller in (43).

\[
L_{LQ} = (J_H^T J_H + \beta I_m)^{-1} J_H^T, \quad \beta > 0
\]
\[
= J_H^T (J_H^T J_H + \beta I_n)^{-1}, \quad \beta > 0.
\] (43)

After utilizing (4), we can conclude that the LQ optimal controller corresponds to an \( \ell_2 \) controller with \( X_1 \) given by (27), \( \gamma = 1 \), and \( W = I_m \). With the control structure in [20] based on Figure 2, their ILC controlled system is not asymptotically stable for the case \( p < \min(m, n) \), Lemma 1.

5. EXPERIMENTAL COMPARISON

In this section we compare the \( \ell_2 \) and \( \ell_\infty \) Hankel ILC control strategies with the LQ optimal control strategy, (43), on a two-mass system depicted in Figure 5. The system is driven by a DC motor which is connected to the first mass \( M_1 \). The second mass \( M_2 \) is connected to the first through a flexible shaft.

The model of the two-mass system, Figure 6, includes the inertia of the masses and the flexibility and damping of the shaft, but leaves out cogging effects of the motor and the small amount of friction in the bearings. The sensor attached to the second mass has a resolution of \( \pi \cdot 10^{-3} \) radians, and the system is sampled with a frequency of \( f_s = 1000 \) Hz.

Since without feedback controller \( C(z) \) the system is marginally stable, we choose to first stabilize the system using feedback control (in time domain). The impulse response of model \( J(z) = (1 + P(z)C(z))^{-1} P(z) \) is given in Figure 7.
Figure 6. Model $P(z)$ of the Two-mass setup.

Figure 7. Impulse response of $J(z)$.

The reference signal $r_{ref}$ and initial command signal $f_{init}$ are presented in Figure 8. Since we are dealing with a closed loop system in time domain, the exogenous signal $r - y_{init}$ equals the initial error signal $(1 + P(z)C(z))^{-1}(r_{ref} - P(z)f_{init})$.

From Figure 8, it can be seen that only a small interval of the point-to-point motion is used for actuation (58 samples) while the observation interval contains 443 samples, hence $J_H \in \mathbb{R}^{443 \times 58}$. The order of $J(z)$ is 7, resulting in $\text{rank}(J_H) = p = 7$, $J_R \in \mathbb{R}^{7 \times 58}$, and $J_L \in \mathbb{R}^{443 \times 7}$. This results in $J_R = \Sigma_1 V_1^T$ and $J_L = U_1$.

Figure 8. Reference signal $r_{ref}$ and initial command signal $f_{init}$. 
The $L_L$ is designed to achieve optimal performance, i.e., $L_L = J_L^t = U_1^T$, and $X_1$ is based on (27). The weighting matrix $W$ for the $\ell_2$ controller is chosen diagonal. Initially, we calculated the theoretically converged command signal $f_\infty$ given by (34). Subsequently, we used it to determine the entries in $W$

$$W(i, i) = \alpha |f_\infty(m_1 + i - 1)| + 1, i = 1, 2, \ldots, m,$$

with $\alpha = 500$. By defining $W$ as in (44), the command signal is penalized most at those time instants where theoretically $f_\infty$ has the largest amplitude. Note that this is just one possible choice for $W$.

The experimental results for LQ optimal, $\ell_2$, and $\ell_\infty$ control strategies are given in Figures 9, 10, 11, and 12 respectively. For these experiments, we have set $\beta = 0.1$ and $\gamma = 0.5$. The initial residual vibration $\tilde{r}$ during $k = 0$ is shown in Figure 9. The error norms $||e_k||$ as function of trial number $k$ are presented in Figure 10. It can be seen that the norms converge monotonically up to approximately -25dB. For $||e_k|| < -25$dB, the signals are dominated by sensor noise (the finite resolution of the encoder). The error signals during the observation time interval are presented in Figure 11. After convergence, the measured errors are in-between the optimal errors presented in the left three plots of Figure 11, and the “worst case” steady state errors in the right three plots. Recalling that the resolution of the encoder equals $\pi \cdot 10^{-3}$ rad, even the worst case errors are within one encoder count.

![Figure 9. Residual vibration during observation interval without Hankel ILC application (k = 0).](image)

![Figure 10. Error norm as function of trial k. Left: LQ optimal Hankel ILC, Center: $\ell_2$ Hankel ILC, Right: $\ell_\infty$ Hankel ILC.](image)
Figure 11. Error signals during observation interval. Top: LQ optimal Hankel ILC, Center: $\ell_2$ Hankel ILC, Bottom: $\ell_\infty$ Hankel ILC. Left, top to bottom: optimal error signals after convergence (trials $k = 50$, $k = 47$, and $k = 48$ respectively), Right, top to bottom: worst case error signals after convergence (trials $k = 48$, $k = 50$, and $k = 50$ respectively).

Though the error signals for all three control strategies look similar, the command signals clearly differ. In the top plot of Figure 12, the converged command signals for LQ optimal control and $\ell_2$ control are shown. Due to the chosen $W$, the $\ell_2$ command signal has a maximum amplitude which is 5 percent smaller than the LQ optimal command signal. In the bottom plot, LQ optimal control is compared to $\ell_\infty$ control. In this case, the reduction in maximum amplitude is 18 percent.

Figure 12. Converged command signals for LQ optimal, $\ell_2$ and $\ell_\infty$ Hankel ILC.
6. CONCLUDING REMARKS

In this paper, we have derived a special form of Iterative Learning Control (ILC), denoted by Hankel ILC, which is capable of suppressing residual vibrations in flexible systems performing point-to-point motions. This special form has been obtained by introducing an actuation and observation time window in the ILC control scheme. As a result, the time interval during a trial in which the system is actuated is separated from the time interval in which the measured output values used in ILC are collected.

Within a newly proposed ILC control structure, we have shown that optimal performance and asymptotic stability of the ILC controlled system can be associated with different parts of the ILC controller. Furthermore, we have proven that there is the additional freedom in Hankel ILC to manipulate the amplitude of the command signal, and we have illustrated this with experiments on a flexible motion system.

From an input shaping point of view, Hankel ILC can be considered a new input shaping technique capable of iteratively suppressing residual vibrations originating from multiple modes in presence of model uncertainty. From an ILC point-of-view, introduction of time windows in the ILC framework has dramatically extended the range of applications for ILC: residual vibration suppression in point-to-point motions is just one of the many possible uses of time windows in ILC. For an illustration of the freedom of choice of time windows, see [23].

Next to the introduction of time windows in ILC, the proposed ILC control structure has given new insights in the design freedom of ILC controllers which have not been mentioned before in ILC literature.

In this paper, we have assumed that the plant model and true system coincide. However, depending on the application, model uncertainty can be a serious issue which can not be neglected. Within the Hankel ILC framework, further research could be performed to analyze the effects of model uncertainty on robust stability, robust performance, and the possible performance degradation as a result of command signal manipulation. With these effects known, Hankel ILC control design might be adapted, e.g., by introducing Higher Order ILC, [29], or including Current Cycle Feedback ILC, [30], to increase robustness of the ILC controlled system against model uncertainty.

Further research could also be sought in the exploitation of time windows, thereby extending the work reported in [23].

REFERENCES


