Mathematical Growth Model for Aneurysms

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1 General Introduction

This report is concerned with the modeling of the growth of aneurysms. They are not rare and their incidence tends to rise with age. This phenomenon is a major problem, because of the possibility of rupture, causing bleeding in the surrounding tissue. Quantification of this possibility becomes important to decide which patient should be surgically treated. Therefore more information is required about the onset and development of the aneurysm and eventually rupture of the vessel wall.

Arterial aneurysms are localized dilatations of the blood vessel wall and are mainly concentrated in the large vessels of the human body [13]. Blood vessel walls are composed of three distinct layers or tunics. These layers surround a central blood-containing space, the vessel lumen. The innermost layer is the tunica intima that contains the endothelium and connective tissue. Its main role is to be a selective barrier between the blood and the surrounding tissues and to be a wall shear stress sensor. [1], [13]

The middle layer, the tunica media, is mostly circularly arranged smooth muscle cells and sheets of elastin. Depending on the needs of the body either vasoconstriction (reduction in lumen diameter due to smooth muscle contraction) or vasodilatation (widening of the lumen due to smooth muscle relaxation) can be effected. Because small changes in blood vessel diameter greatly influence blood flow and blood pressure, the activities of the media are critical in regulating circulatory dynamics. Generally, the media is the largest layer in arteries, which bear the greatest responsibility for maintaining blood pressure and continuous blood circulation. [13].

The outermost layer of a blood vessel wall is the tunica adventitia. This layer is composed largely of collagen fibers that protect and reinforce the blood vessel and anchor it to surrounding structures [13].

Intravascular pressure and a weakening in the arterial vessel wall lead to the formation of aneurysms. It is thought that the primary cause is a weakening of the tunica media due to degradation of elastin. This degradation does not lead to wall rupture directly. Instead the increase in stresses inside the vessel wall due to this weakening causes remodeling of the tissue by proliferation of collagen fibers that will lead to a stiffer vessel wall [9], [13].

Growth and remodeling are fundamental mechanical processes both in the normal development of tissues and in several pathological conditions. Mechanical quantities like stress and strain in the tissue can modulate its growth [10]. The mechanics of aneurysm pathogenesis are not well yet understood. Although different biological mechanisms are identified as possible reasons for the genesis, development and eventually rupture of the aneurysm, a fully descriptive theory that explains the relevant biomechanical mechanisms can-
not be found in literature [1]. A distinction can be made between models based on structural changes in the tissue without any volume change and approaches based on volumetric growth. The first model attributes the evolution of an aneurysm to a consequence of the remodeling of the artery’s material constituents [13]. A volumetric growth model is based on the change of a reference state. Rodriguez et. al. (1994) has proposed a volumetric growth model demonstrating the importance of a residual stress field. Residual stress results from local incompatible growth of the tissue. The overall goal of this report is to suggest a new approach to describe mechanically dependent growth and remodeling of the tissue. This report introduces a simple volumetric growth model by introducing an extra variable to the compressible Neo-Hookean constitutive law with emphasis on numerical implementation rather than physiological realism.

Chapter 2 discusses simple (in)compressible isotropic hyper-elastic material laws, like Neo-Hookean and Mooney-Rivlin behavior. They are evaluated in the finite element method (FEM) package Sepran verifying the correctness of the implementation of this behavior. Chapter 3 introduces the growth model and discusses various simulations that are performed. Finally a conclusion and discussion will lead to recommendations for future research.
2 Hyper-elastic materials

2.1 Introduction

This section discusses simple isotropic hyper-elastic material laws that are used for modeling the mechanics of biological tissues. The characteristic of this kind of materials is that all the energy that is put into the system during loading can be regained when the stress is released. That means that no (elastic) energy is lost in the form of heat.

Simple uniaxial extension and compression tests are carried out with the FEM package Sepran and compared to the analytical solution to verify the correctness of the models. First incompressibility is assumed, which is the case for many biological tissues. However, remodeling of the tissue due to degradation of one or more of its constituents causes tissue growth. This leads to compressible behavior and modeling of this behavior becomes important.

2.2 Materials and methods

2.2.1 Mathematical model

In this section nonlinear constitutive laws are considered in three dimensional space. If body forces are neglected the equation of conservation of momentum and the incompressibility condition for incompressible behavior can be written as

\[
\begin{align*}
\nabla \cdot \sigma &= 0 \quad \text{in } \Omega(t) \\
\det(F) - 1 &= 0 \quad \text{in } \Omega(t)
\end{align*}
\]

with \( \sigma \) the Cauchy stress tensor and \( F = (\nabla_0 x)^T \) the deformation gradient tensor defining the deformation from the reference state \( \Omega_0 \) with position vectors \( x_0 \) to the current state \( \Omega(t) \) with position vectors \( x(t) \). The Cauchy stress can be written as

\[
\sigma = -pI + \tau
\]

(2)

where \( p \) is the hydrostatic pressure and \( \tau \) the deviatoric stress resulting from deformation. The Cauchy stress in this kind of material laws can be described with a so called \textit{Energy Density Function} (EDF) [6].

\[
\sigma = -pI + \tau = -pI + \frac{2}{J} F \cdot \frac{\partial E}{\partial C} \cdot F^T
\]

(3)

with \( E \) the strain energy, \( J \) the determinant of \( F \) and \( C \) the right Cauchy-Green stress tensor.
In case of an isotropic material the strain energy only depends on the invariants of the tensor $C$ \[6\]. Due to the objectivity constraint the left Cauchy-Green or Finger tensor $B$ is used instead of $C$.

The considered incompressible constitutive models are Neo-Hookean, suitable for deformations up to 10% and Mooney-Rivlin that describes larger deformations. The stress-strain relations are

\[
\text{Neo-Hookean: } \sigma = -pI + 2C_{01}B \quad (4)
\]

\[
\text{Mooney-Rivlin: } \sigma = -pI + 2 \left( C_{01} + C_{02}tr(B) \right)B - C_{02}B^2 \quad (5)
\]

with $C_{01}$ and $C_{02}$ material coefficients.

In case of compressible behavior the material will change in volume and therefore $J = \det(F) \neq 1$. The equation of conservation of momentum can be written as

\[
\nabla \cdot \sigma = 0 \quad \text{in } \Omega(t) \quad (6)
\]

The constitutive model that is used is compressible Neo-Hookean and the relation between the stress and strain is

\[
\sigma = K(J-1)I + \frac{G}{J} \left( B - \frac{J^2}{3}I \right) \quad (7)
\]

with $K$ the bulk modulus and $G$ the shear modulus.

### 2.2.2 Model implementation

Sepran is used to solve the equations with an updated Lagrange approach. In figure 1 a schematic representation of describing the deformation within this approach is depicted. The updated Lagrange method uses transformations back to the last known configuration $\Omega_n$. Moreover since the set of equations is nonlinear in terms of displacements Newton iterations are used to solve the equations.
Calculation of the solutions is done for a brick consisting of one 27-node isoparametric hexahedral element using two different implementation methods; SEPCOMP and a so-called main program. The brick has initial volume 1 and the mesh is presented in figure 2.

The element type used to solve equations (1), (4) and (5) is 201 which is a discontinuous pressure element with pressure and pressure gradient at node 14. Element type 200 is used to solve equations (6) and (7). Because pressure do not play a role for compressible behavior only three degrees of freedom are calculated.

The boundary conditions that have to be met to simulate uniaxial tension or compression are (see figure 2)

- suppression of motion of the first dof of $S_5$.
- suppression of motion of the second dof of $S_2$.
- suppression of motion of the third dof of $S_1$.
- prescribed displacements($u_x$) or loads ($\sigma_{xx}$) on $S_3$.

The general boundary element type, element 210, is used for implementation of the natural boundary conditions.
The assumptions made for the above described simulations are presented in tables 1 and 2. The convergence criterion, \( \epsilon \), is defined as \( \frac{u_{\text{iter}}}{u_{\text{incr}}} < \epsilon \). The values for the parameters are not physiological.

Table 1: Assumptions made for incompressible material laws.

<table>
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</tr>
<tr>
<td>( \Delta t )</td>
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</tr>
<tr>
<td>( C_{01} )</td>
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</tr>
<tr>
<td>( C_{02} )</td>
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Table 2: Assumptions made for compressible material laws.

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<td>( \Delta t )</td>
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</tr>
<tr>
<td>( K )</td>
<td>3 \cdot 10^3 [Pa]</td>
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<td></td>
</tr>
<tr>
<td>( G )</td>
<td>1 \cdot 10^3 [Pa]</td>
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2.2.3 Simulations

Simple 3D extension and compression simulations are performed in Sepran for incompressible as well as compressible materials and compared to the analytical solution. For uniaxial tests in the \( x \)-direction the deformation
tensor \((\mathbf{F})\) for incompressible problems becomes
\[
\mathbf{F} = \begin{pmatrix}
\lambda & 0 & 0 \\
0 & \frac{1}{\sqrt{\lambda}} & 0 \\
0 & 0 & \frac{1}{\sqrt{\lambda}}
\end{pmatrix}
\implies
\mathbf{B} = \begin{pmatrix}
\lambda^2 & 0 & 0 \\
0 & \frac{1}{\lambda} & 0 \\
0 & 0 & \frac{1}{\lambda}
\end{pmatrix}
\]
with \(\lambda\) the stretch of the material.

In this case the only stress component that is nonzero is \(\sigma_{xx}\). Because \(\sigma_{yy} = \sigma_{zz} = 0\) the relation between the pressure and the stretch can be found and eventually \(\sigma_{xx}\) becomes

Neo-Hookean:
\[
\sigma_{xx} = 2C_{01} \left( \lambda^2 - \frac{1}{\lambda} \right)
\] (8)

Mooney-Rivlin:
\[
\sigma_{xx} = \frac{2}{\lambda^2} \left( C_{01} \lambda^4 + C_{02} \lambda^3 - C_{01} \lambda - C_{02} \right)
\] (9)

For compressible materials the \(\text{det}(\mathbf{F}) \neq 1\) and therefore the definition of the stretches in the other principal directions cannot be defined using the stretch in the direction of the force. The deformation tensor for uniaxial behavior in x-direction becomes
\[
\mathbf{F} = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_2
\end{pmatrix}
\implies
\mathbf{B} = \begin{pmatrix}
\lambda_1^2 & 0 & 0 \\
0 & \lambda_2^2 & 0 \\
0 & 0 & \lambda_2^2
\end{pmatrix}
\]
with \(\lambda_1\) the stretch of the material in the direction of the force and \(\lambda_2\) the stretch in the other principal directions.

As for incompressible materials the only stress component that is nonzero is \(\sigma_{xx}\). With \(J = \text{det}(\mathbf{F}) = \lambda_1 \lambda_2^2\) the equilibrium equations are
\[
\sigma_{xx} = K(\lambda_1 \lambda_2^2 - 1) + \frac{G}{\lambda_1 \lambda_2^2} \left( \lambda_1^2 - \lambda_1^2 \lambda_2^4 \right) = S
\] (10)
\[
\sigma_{yy} = \sigma_{zz} = K(\lambda_1 \lambda_2^2 - 1) + \frac{G}{\lambda_1 \lambda_2^2} \left( \lambda_2^2 - \lambda_1^2 \lambda_2^4 \right) = 0
\] (11)
with \(S\) the prescribed stress.
Subtraction of equation (10) from equation (11) and rearranging the variables gives eventually a relation between $\lambda_2$ and $\lambda_1$.

$$\lambda_2^2 = \frac{\lambda_1}{\left(\frac{S}{\sigma} + \frac{1}{\lambda_1}\right)}$$  \hspace{1cm} (12)

The substitution of equation (12) into equation (10) will finally lead to a relation between $\lambda_1$ and $S$. Because of the complexity of the resulting equation Matlab (version 7.0.4) is used to derive the analytical solution. Details of this program can be found in the Appendix.

### 2.3 Results

The results for incompressible uniaxial extension and compression simulations are presented in figures 3 and 4. In these figures the difference between the calculated SEPCOMP stress and those from the analytical solutions in x-direction can be seen. For compression or extension of 10% the error between stress of the analytical solution and the calculated solution for incompressible Neo-Hookean behavior is about 2%, but increases rapidly when the stretch increases. Mooney-Rivlin shows the same behavior but the error for compression is 2% and for extension 2.5% for the same loading conditions.

The difference between the uniaxial extension/compression tests of SEPCOMP, the main program and the analytical solution of compressible materials is visualized in figure 5. When using a main program the analytical solution and the calculated solution coincide even for stretches of 50%. The error between the calculated SEPCOMP stress and the analytical stress is 20% for stretches of 10%.
Figure 3: The relation between $\sigma_{xx}$ and $\lambda$ for incompressible Neo-Hookean materials for the SEPCOMP and the analytical solution with $C_{01} = 5 \cdot 10^2$ Pa.

Figure 4: The relation between $\sigma_{xx}$ and $\lambda$ for incompressible Mooney-Rivlin materials for the SEPCOMP and the analytical solution with $C_{01} = 5 \cdot 10^2$ Pa and $C_{02} = 1 \cdot 10^3$ Pa.
2.4 Discussion

3D uniaxial extension and compression tests were used for evaluation of incompressible and compressible constitutive laws that are implemented in FEM package Sepran. A reason for the difference between the analytical and the SEPCOMP solution for the two incompressible constitutive laws could be that calculation of the displacements, stresses and strains fails. Calculation of these variables is done with subroutine deriv. Calculated displacements were not similar to the prescribed displacements so it might be that the essential boundary conditions are not correctly prescribed.

Stretches on the x-axis of figures 3 - 5 on pages 10 and 11 are stretches determined from the prescribed displacements \( \lambda = \frac{l_0 + du}{l_0} = 1 + du \), since \( l_0 = 1 \) instead of the calculated displacements and may present wrong results.

Another problem observed was that the calculated pressure using subroutine deriv was twice as large than the interpolated value. The former seems to be correct depending on the amount of compression or extension. The error between the calculated pressure of the analytical and SEPCOMP solution is 5.1% for stretches of 10% but still unacceptably large.

Compressible Neo-Hookean behavior, only implemented in a main program,
shows a good similarity with the analytical solution. Again SEPCOMP fails to determine the right solution. The only distinction between the two implementation methods are the boundary conditions; SEPCOMP uses prescribed displacements and the main program uses prescribed stresses. It is more likely that this difference in solution arises from difference in implementation method instead of different boundary conditions. Namely, boundary conditions are coupled. If on a specific surface displacements are prescribed, stresses are unknown and vice versa.

2.5 Conclusion

It can be concluded that solutions of incompressible Neo-Hookean and incompressible Mooney-Rivlin constitutive laws are not similar to those of the analytical solution when SEPCOMP is utilized. However, due to lack of time it is not verified if the same results are obtained if a main program is used. Compressible Neo-Hookean material is implemented well and show reliable results comparing with the analytical solution. Therefore this material law can be useful for modeling growth of aneurysms.
3 Growth Model

3.1 Introduction

Every time the tissue is exposed to an external load, in this case the blood pressure, it causes internal strains. These strains can be larger or smaller in weakened regions compared to healthy regions depending on the magnitude of the pressure. The tissue, however, wants to optimize its mechanical loading and the primary reaction is proliferation/breakdown of collagen leading to a more/less stiffer vessel wall to compensate for the increase/decrease.

Different approaches of describing mechanically dependent adaptation of the tissue are identified. A model based on structural changes in the tissue without any volume change attributes the evolution of an aneurysm to a consequence of the remodeling of the artery’s material constituents [13]. Approaches based on volumetric growth like the one Rodriguez et. al. (1994) had proposed demonstrates the importance of a residual stress field. Residual stress results from local incompatible growth of the tissue.

The overall goal of this report is to suggest a new approach to describe mechanically dependent growth and remodeling of the tissue. This report introduces a simple volumetric growth model by introducing an extra variable to the compressible Neo-Hookean constitutive law with emphasis on numerical implementation rather than physiological realism.

3.2 Materials and methods

3.2.1 Mathematical model

It is believed that there exists a relation between mass change and strain. A simple linear relation is supposed (figure 6). If the strains are in a so called dead zone no mass change will evolve. Exceeding this zone the tissue will be triggered and starts to remodel. The mass of the tissue will either decrease or increase depending on the strain. This relation is presented in equation 13.

\[
\frac{dm}{dt} = \begin{cases} 
  m\alpha (\epsilon - \epsilon_{\text{opt}}) & -\infty < \epsilon < (\epsilon_{\text{opt}} - \delta) \\
  0 & (\epsilon_{\text{opt}} - \delta) < \epsilon < (\epsilon_{\text{opt}} + \delta) \\
  m\alpha (\epsilon - \epsilon_{\text{opt}}) & (\epsilon_{\text{opt}} + \delta) < \epsilon < \infty 
\end{cases}
\]

(13)

with \( \frac{dm}{dt} \) the mass change in time, \( m \) the mass of the tissue, \( \alpha \) the slope, \( \epsilon \) the relevant strain component, \( \epsilon_{\text{opt}} \) the optimal strain and \( \delta \) the width of the dead zone.

Calculation of the mass change and therefore the volume change, assuming
a constant density ($\rho$), is straightforward.

$$\frac{dV}{dt} = \frac{1}{\rho} \frac{dm}{dt} = \frac{m}{\rho} \alpha (\epsilon - \epsilon_{opt}) = V \alpha (\epsilon - \epsilon_{opt})$$ \hspace{1cm} (14)

To describe growth compressible material laws are a good basis. During this project compressible Neo-Hookean (equation (15)) is used, because this law describes large deformations well, as discussed in section 2.

$$\sigma = K(J - 1)I + \frac{G}{J} \left( B - J^{\frac{2}{3}}I \right)$$ \hspace{1cm} (15)

The first term of the constitutive law, $K(J - 1)I$, describes the influence on the stresses due to the volume change and the second term $B - J^{\frac{2}{3}}I$ due to deformation of the tissue.

Implementation of the relation between volume change and strain is done by modification of the constitutive law. An extra variable, $J_{growth}$, has been added to equation (7).

$$\sigma = K(J - J_{growth})I + \frac{G}{J} \left( B - J^{\frac{2}{3}}I \right)$$ \hspace{1cm} (16)

If there is no volume change $J_{growth}$ is 1, otherwise $J_{growth}$ has a certain value depending on the stimulus. Using an Euler integration scheme the
new value of $J_{\text{growth}}$ after time $\Delta t$ can be computed.

\[ J_{\text{growth}}^{n+1} = 1 + \frac{1}{\rho} \left( \frac{dm}{dt} \right)^n \Delta t \]  

(17)

### 3.2.2 Model implementation

There are several possibilities for implementation of the growth model, but only one is actually used for simulations. The important steps are:

**Method I:**

1. mechanical loading with $J_{\text{growth}} = 1$ and $\sigma_{\text{ext}} \neq 0$: $\vec{x}_0 \rightarrow \vec{x}_m$
   
   (a) if $\| \epsilon^{\text{node}} - \epsilon_{\text{opt}} \| \leq \delta \Rightarrow \text{stop}$
   
   (b) if $\| \epsilon^{\text{node}} - \epsilon_{\text{opt}} \| > \delta \Rightarrow \text{go to step 2}$

2. $J_{\text{growth}}^{\text{new}} = 1 + \frac{1}{\rho} \left( \frac{dm}{dt} \right)^n \Delta t$

3. simulate growth with $\sigma_{\text{ext}} = 0$ and $J = J_{\text{growth}}^{\text{new}}$: $\vec{x}_0 \rightarrow \vec{x}_g$

4. update the mesh: $\vec{x}_0^{\text{new}} = \vec{x}_g$

5. go to step 1

The first step is mechanical loading of the tissue for a certain value of $\sigma_{\text{ext}} \neq 0$ and $J_{\text{growth}} = 1$ resulting in a configuration with displacements $\vec{x}_m$. Every node of the geometry has to satisfy the inequality, so the program only stops when all the nodes have a strain less than $\epsilon_{\text{opt}} + \delta$. If this is not the case $J_{\text{growth}}^{\text{new}}$ is calculated.

Step 3 represents the growth step. Important is that this growth step is applied on the initial configuration. No external load is applied so only internal strains, due to a deviation of $J_{\text{growth}}$ from one, cause new displacements $\vec{x}_g$. These displacements are used to update the unloaded reference configuration and the process starts again until an equilibrium is reached with a new, changed, volume.

The load $\sigma_{\text{ext}}$ to be applied in the grown state needs some discussion. If the load corresponds to a surface load, expressed in $N/m^2$, such as in case of blood pressure acting on the inside of the vessel wall, $\sigma_{\text{ext}}$ may be left unchanged. If, however, the load corresponds to a surface force, expressed in $N$, $\sigma_{\text{ext}}$ has to be corrected for the change in surface area, induced in the growth step.
3.2.3 Simulations

First remodeling of a brick for homogeneous deformation (figure 7a) is considered evaluating if the model is suitable to describe growth. Therefore a simple 3D uniaxial extension test is carried out. The mechanics of uniaxial extension tests are described in section 2.2.3 on page 7.

Thereafter material is used to simulate inhomogeneous deformation (figure 7b) testing the model for varying number of elements, different loads and different values for the width of the dead zone ($\delta$) These parameters will all have a distinct influence on the solution of the equations.

For both deformations the general strain parameter $\epsilon$ used for comparison between the optimal strain is defined as $\epsilon_{xx}$ since loading is applied in $x$-direction.

![Figure 7: The initial mesh for homogeneous and inhomogeneous deformation.](image)

During the simulations one parameter is varied while keeping the other at a constant value. Values of the above described parameters for the different simulations are presented in table 3.

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<td>1</td>
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<td>0.01-0.05</td>
<td>0.10</td>
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Calculation of the solution using Sepran is done again using isoparametric hexahedral elements of 27 nodes. The element type used is 200 because pressure do not play a role in compressible behavior. Every node in this element has three degrees of freedom that are calculated during the simulations. The boundary conditions that have to be satisfied are the same (see...
section 2.2.2) and the assumptions made for the simulations are summed in table 4. The convergence criterion, $\epsilon$, is defined as $|\frac{u_{\text{iter}}}{u_{\text{incr}}}| < \epsilon$.

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<tr>
<td>$G$</td>
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3.3 Results

3.3.1 Homogeneous

First the brick of initial volume one for homogeneous deformation is considered. The red line in figure 8 represents the axial strain during the growth process for each node of the geometry. Because the strain in all nodes will be equal in all growth loops only one line is seen. During the growth process the strain in x-direction is decreasing. Eventually a new equilibrium is reached with an axial strain within the dead zone and an increased volume (figure 9).

3.3.2 Inhomogeneous

Simulation 1: Varying the number of elements

The difference between the axial strain using different number of elements is depicted in figures 10 and 11. Each line (red) represents the axial strain of a node of the geometry during each growth loop. The obtained results show that no difference in axial strains during the growth process and after convergence is observed between 1 element and 8 elements. The initial and final meshes are presented in figures 12 and 13.

Simulation 2: Differing the applied load

The influence of the applied load on the growth process and the final volumes are presented in figures 14 and 17. Only the results of two different
loads are presented. When applying a small load the axial strains after the first mechanical loop are less than the optimal strain. Decreasing the volume results in an increase in axial strain and finally a new equilibrium is reached. The opposite occurs when a large load is applied on the surface.

**Simulation 3: Differing the width of the dead zone ($\delta$)**
The width of the dead zone indicates the variation in strains that will not lead to tissue growth. Results of the influence of this variable are presented in figures 19 - 21. Because strain variation is less for a small $\delta$ more growth loops are needed to get convergence. The final volume is larger what is expected according to the assumed relation between mass change and strain. A larger value of $\delta$ leads to less mass increase and more strain variation.

![Graph](image)

Figure 8: Homogeneous deformation: The axial strains induced by an external load of 500 N during the growth process. # elements is 1, $\delta = 0.05$ and $\epsilon_{opt} = 0.10$. After 3 growth loops convergence is reached.
Figure 9: Homogeneous deformation: The initial mesh (blue) and the final changed mesh (red) after the last growth loop.

Figure 10: Influence of varying number of elements (inhomogeneous deformation): The axial strains induced by an external load of 500 N during the growth process. # elements is 1, $\delta = 0.05$ and $\epsilon_{opt} = 0.10$. After 4 growth loops convergence is reached.
Figure 11: Influence of varying number of elements (inhomogeneous deformation): The axial strains induced by an external load of 500 N during the growth process. # elements is 8, δ = 0.05 and ϵ_{opt} = 0.10. After 4 growth loops convergence is reached.

Figure 12: Influence of varying number of elements (inhomogeneous deformation): The initial mesh (blue) and the final changed mesh (red) after the last growth loop for 1 element.
Figure 13: Influence of varying number of elements (inhomogeneous deformation): The initial mesh (blue) and the final changed mesh (red) after the last growth loop for 8 elements.

Figure 14: Influence of different loads (inhomogeneous deformation): The axial strains induced by an external load of 50 N during the growth process. # elements is 1, δ = 0.05 and ε_{opt} = 0.10. After 11 growth loops convergence is reached.
Figure 15: Influence of different loads (inhomogeneous deformation): The initial mesh (blue) and the final changed mesh (red) after the last growth loop for $F = 50$N.

Figure 16: Influence of different loads (inhomogeneous deformation): The axial strains induced by an external load of 500 N during the growth process. $\#$ elements is 1, $\delta = 0.05$ and $\varepsilon_{opt} = 0.10$. After 4 growth loops convergence is reached.
Figure 17: Influence of different loads (inhomogeneous deformation): The initial mesh (blue) and the final changed mesh (red) after the last growth loop for $F = 500\text{N}$.

Figure 18: Influence of varying the width of the dead zone (inhomogeneous deformation): The axial strains induced by an external load of 500 N during the growth process. # elements is 1, $\delta = 0.01$ and $\epsilon_{opt} = 0.10$. After 4 growth loops convergence is reached.
Figure 19: Influence of varying the width of the dead zone (inhomogeneous deformation): The axial strains induced by an external load of 500 N during the growth process. # elements is 1, $\delta = 0.05$ and $\epsilon_{opt} = 0.10$. After 4 growth loops convergence is reached.

Figure 20: Influence of varying the width of the dead zone (inhomogeneous deformation): The initial mesh (blue) and the final changed mesh (red) after the last growth loop for $\delta = 0.01$. 
3 Discussion

3.4.1 Growth model

In section 3.2 the relation between mass change and strain is introduced. It is believed that there exists such relation but the exact definition is not yet clear, but a linear relation seems a good starting point.

As mentioned before only method I is implemented. The growth loop is entered when the strains are larger than $\epsilon_{opt} + \delta$ and is calculated with $\sigma_{ext} = 0$, but this is not physiological. The vessel wall always senses the blood pressure and responds to it and therefore mechanical loading and growth cannot be split up in two processes. Other implementation methods that combine these processes are more likely and should produce better physiological results.

3.4.2 Assumptions

The geometry represents not a normal vessel wall but a brick of massif tissue. The values for the parameters like the applied force, shear modulus, bulk modulus are not physiological but arbitrarily chosen. In order to obtain results that are also suitable for quantitative analysis realistic input parameters are needed.

Another point of issue is the strain parameter, $\epsilon_{node}^{\text{node}}$, that is used during this project to solve the inequality $\| \epsilon_{node}^{\text{node}} - \epsilon_{opt} \| \leq \delta$. The strain in x-
direction was chosen, because the load was applied in the same direction. The tissue however does not sense the direction in which it is stretched and therefore a strain value used for solving the inequality should be orientation independent as well.

### 3.4.3 Results

The first simulation was done for a homogeneous brick of volume one and it can be concluded that the growth model gives expected results.

Three different simulations are performed in order to obtain more insight in the quality of the model. Only qualitative results are presented because no extensive analysis is done. The number of elements do not lead to more accurate results, because the interpolation scheme is linear. More elements caused errors during the simulation. When using more complex and larger geometries the number of elements might greatly enhance the accuracy.

Different loads will lead to different final configurations as one could see in the figures 15 and 17. This is what is expected, because the tissue wants to optimize its mechanical loading and responds to larger strains by proliferation of the constituents and to smaller strains by breakdown.

The influence of the width of the dead zone is straightforward. Small values of $\delta$ result in more growth loops and a slightly less variation in strains and the opposite occurs for larger values of $\delta$.

A common observation is made about the final mesh in the inhomogeneous case. The left upper corner tends to shift in the negative $y$-direction. It seems to be that shearing occurs but a uniaxial extension test is performed. More elements does not solve this problem. One reason could be that the normal of surface $S_3$ is changed and therefore the material exert shear as the direction of the force remains in the $x$-direction. Another reason could be that it has to do something with the boundary conditions.
3.5 Conclusion and Recommendations

Modifying the existing compressible Neo-Hookean constitutive law by adding a growth stimulus represents a good basis of describing growth of aneurysms. Unconfined uniaxial elongation exhibit the capability of the developed model to describe growth. These tests reveal the influence of several parameters that are important for the quantity of volume increase or decrease.

The results presented in this report are obtained for arbitrary input parameters and can only reflect a qualitative behavior of tissue remodeling. Quantitative analysis is only useful when physiological values for the parameters are used. Not only the parameters have to be physiological the geometry have to have a cylindrical shape with a vessel lumen surrounded by a vessel wall of certain thickness and contain information about the individual constituents (elastin and collagen).

And of course more information is acquired on the relation between strain and mass change to get a better understanding about the coupling of these two variables.
4 Appendix

Matlab program for calculation of the analytical solution of compressible Neo-Hookean material.

```matlab
%%
%% program compNH.m
%%
%% This program uses the analytical solution to find the relation
%% between the stress and the stretch of a compressible
%% Neo-Hookean material. The analytical solution is:
%%
%% \[ \sigma = K \times (L1 \times L2^2 - 1) + \left( \frac{G}{L1 \times L2^2} \right) \times \left( L1^2 - (L1 \times L2^2)^{(2/3)} \right) \]
%%
%% 0 = K \times (L1 \times L2^2 - 1) + \left( \frac{G}{L1 \times L2^2} \right) \times \left( L1^2 - (L1 \times L2^2)^{(2/3)} \right) - \sigma
%%
%% with \( L2^2 = L1/((\sigma/G) + 1/L1) \)
%%

clear all;
close all;
cd;

syms L1 S;

G = input('Give the value for the shear modulus: ');  
if (G <= 0) 
    disp('********************************'); 
    disp('G should be larger than zero!!!!'); 
    disp('********************************');  
    G = input('Give the value for the shear modulus: ');  
end

K = input('Give the value for the bulk modulus: ');  
if (K <= 0) 
    disp('********************************'); 
    disp('K should be larger than zero!!!!'); 
    disp('********************************');  
    K = input('Give the value for the bulk modulus: ');  
end

% The function 'f' is the equation to be solved. \( L2^2 \)
% is included in 'f' so that only an equation remains
% that is dependent on L1 and sigma (S)
% \[ f = K(L1^2/(S/G + 1/L1) - 1) + G/L1^2 \times (S/G + 1/L1) \times (L1^2 - (L1^2/(S/G + 1/L1))^{(2/3)}) - \sigma; \]

% colom is the applied stretch L1
colom = 0.05:0.05:2.5;
h = 0;
g = 0;
stress = [];
for i = 1:length(colom)  
h = subs(f,L1,colom(i)); 
g = eval(solve(h)); 
stress(i) = max(g);  
end

% For every value of i the stretch is included
% For every value of i the equation \( f = 0 \) is solved
% Because you get two values the largest one is taken
% (this is the most doubtful part of this m-file,
```
% but if you take the minimum you get no reliable % results)

figure;
plot(colom, stress,'r');
grid;
xlabel('L1 [-]');
ylabel('S [Pa]');
title('Relation between the stretch and the stress.');
xlim([min(colom) max(colom)]);

L2 = [];
for j = 1:length(colom)
    L2(j) = sqrt(colom(j)./(stress(j)/G)+1./colom(j));
end

figure;
plot(colom, L2);
grid;
xlabel('L1 [-]');
ylabel('L2 [-]');
title('Relation between L1 and L2.');
xlim([min(colom) max(colom)]);
References


