CONTROL OF NONLINEAR SYSTEMS WITH TIME-DELAY USING STATE PREDICTOR BASED ON SYNCHRONIZATION

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Abstract
This paper proposes a controller design method for nonlinear systems with time-delay. The method consists of a state feedback and a state predictor based on synchronization of coupled systems. We consider the robust convergency of the prediction error under perturbation and model uncertainties. Then combining the state predictor with a static feedback, we proposed a predictor-based control of nonlinear systems with time-delay at the input. The effectiveness of the proposed method is illustrated by a numerical simulation example.

Key words
nonlinear systems, time-delay, synchronization, state predictor

1 Introduction
Recently the study of time-delay systems has attracted a growing interest in various fields, such as mathematics, information technology, applied physics, biology and social science. Of course, control of time-delay systems has continued to be a hot topic in control engineering and science. In particular, a large number of useful and important results have been obtained for linear time-delay systems (Smith, 1959; Morse, 1976; Sonntag, 1976; Manitius and Olbrot, 1979; Mahmoud, 2000; Boukas and Liu, 2001; Gu et al., 2003). While in practice most systems contain nonlinear elements and are described as nonlinear time-delay systems, the study of control theory for such systems is in progress (Kravaris and Wright, 1989; Huang and Wang, 1992; Oguchi et al., 1994; Oguchi and Watanabe, 1998; Oguchi et al., 2002; Germani et al., 1996; Moog et al., 2000). On the other hand, in the interdisciplinary fields including applied physics, mathematical biology and control science, the synchronization of coupled chaotic systems has been studied from both theory and experiments (Ogorzalek, 1993; Pyragas, 1998; Nijmeijer and Mareels, 1997; Nijmeijer, 2001). As an extension of the synchronization of coupled systems, Voss (Voss, 2000) discovered the occurrence of anticipated synchronization of chaotic systems with time-delay in a unidirectional coupling configuration. The anticipated synchronization can be recognized as a state prediction method which does not require numerical integration. These phenomena are also observed for physical systems. Voss (Voss, 2002) showed an experimental realization of such a phenomenon in an electronic circuit. Masoller (Masoller, 2001) has considered the anticipating synchronization of chaotic external-cavity semiconductor lasers with numerical simulation. Sivaprakasam et al. (Sivaprakasam et al., 2001) have observed the occurrence of anticipating synchronization using chaotic semiconductor diode lasers. These chaotic systems are described by one dimensional systems with time-delay in the state, but the anticipated synchronization can occur in more general systems.

In this paper, we consider control of nonlinear systems with time-delay, and we propose a control design method of nonlinear systems with time-delay using a state prediction based on synchronization. The state prediction scheme based on synchronization (Oguchi and Nijmeijer, 2004; Oguchi and Nijmeijer, 2005) has been developed by the authors in order to predict the behaviour of complex systems with time-delay and it is designed by using delay-dependent stabilizability conditions based on a Lyapunov-Krasovskii functional. This method can be considered as an extension of the anticipating synchronization of chaotic systems proposed in (Voss, 2000). In this paper, we show the robust convergency of the synchronized predictor under perturbations and model uncertainties. First, we derive a sufficient condition for the zero solution of retarded nonlinear systems with perturbations to be asymptotically stable by using the Lyapunov-Razumikhin approach. Then, by using the obtained conditions, we show that the proposed predic-
tion scheme works well even if the system has model uncertainties, perturbations and nonlinear terms which are not contained in the predictor. In addition, we propose a controller design method of nonlinear systems with time-delays by using the proposed predictor. Finally, we show the effectiveness of the proposed controller by a numerical simulation of a boiler system.

2 Predictor based on Synchronization

In this section, we review the state predictor based on synchronization of coupled systems. In the first literature concerning the anticipating synchronization (Voss, 2000), Voss considers the following coupled Ikeda equation,

\[
\begin{align*}
\dot{z}(t) &= -\alpha z(t) - \beta \sin x(t) - \gamma z(t) - \beta \sin x(t), \\
\dot{x}(t) &= -\alpha x(t) - \beta \sin x(t) + y(t).
\end{align*}
\]

The dynamics of the prediction error \(e(t) = z(t - \tau) - x(t)\) is given by \(\dot{e}(t) = -\alpha e(t)\), and a necessary and sufficient condition for the error \(e\) to converge to 0 is that \(\alpha > 0\). This idea can be generalized into multidimensional systems as follows. Consider the following multi-dimensional difference-differential equation system,

\[
\begin{align*}
\dot{z}(t) &= A_0 z(t) + F(y(t)), \\
\dot{y}(t) &= h(x(t)).
\end{align*}
\]  

where \(x \in \mathbb{R}^n\), \(y \in \mathbb{R}^m\), \(A_0 \in \mathbb{R}^{n \times n}\), \(F : \mathbb{R}^m \to \mathbb{R}^n\) and \(h : \mathbb{R}^n \to \mathbb{R}^m\). For the system (1), we consider the following prediction scheme,

\[
\dot{z}(t) = A_0 z(t) + F(y(t))
\]  

in which \(z \in \mathbb{R}^n\). Since the dynamics of the prediction error \(e(t) = z(t - \tau) - x(t)\) is described by

\[
\dot{e}(t) = A_0 e(t),
\]

the error converges to zero if and only if matrix \(A_0\) is Hurwitz, i.e. \(A_0\) has all eigenvalues in the open left half plane. Therefore if \(A_0\) is a Hurwitz matrix, the output \(z(t)\) of the predictor (2) estimates the future value \(x(t + \tau)\) of the state \(x(t)\).

This prediction method is simple and the convergency of the error dynamics is globally guaranteed, while the master system is required to satisfy the conditions

(i) \(A_0\) is a Hurwitz matrix, and
(ii) the time-delayed terms in the dynamics (1) depend on the output only.

These restrictions can be relaxed by adding a coupling term into the slave system as follows.

We consider the following possibly chaotic or complex system

\[
\begin{align*}
\dot{x}(t) &= A_0 x(t) + A_1 x(t - \tau) + f(x(t)) \\
y(t) &= C x(t) \\
\dot{z}(t) &= \varphi(t), \quad t \in [-\tau, 0]
\end{align*}
\]  

where \(x \in \mathbb{R}^n\), \(A_0\) and \(A_1\) are \(n \times n\) matrices, and \(f : \mathbb{R}^n \to \mathbb{R}^n\) is a smooth vector field.

For the system (3), we construct the predictor

\[
\begin{align*}
\dot{z}(t) &= A_0 z(t) + A_1 z(t - \tau) + f(z(t)) + K (C z(t - \tau) - y(t)) \\
z(t) &= z_0, \quad t \in [-\tau, 0]
\end{align*}
\]  

where \(z \in \mathbb{R}^n\) and \(K \in \mathbb{R}^{n \times m}\) is a constant matrix.

For the sake of simplicity of notations, we will often suppress the notation of the explicit dependence of time \(t\) when no confusion can arise. The configuration of the prediction scheme is shown in Figure 1.

\[
\begin{align*}
\Sigma_M : \quad &\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau) + f(x(t)) \\
y(t) = C x(t) \\
\dot{z}(t) = \varphi(t), \quad t \in [-\tau, 0]
\end{align*}
\]  

Then the dynamics of prediction error \(e \triangleq z(t) - x(t)\), where \(z(t) \triangleq z(t - \tau)\), is given by

\[
\begin{align*}
\dot{e} &= A_0 e + (A_1 + KC) e_{\tau} + f(e + x) - f(x) \\
&\triangleq A_0 e + A_1 e_{\tau} + \phi(x, e)
\end{align*}
\]

with \(\phi(x, e) = f(e + x) - f(x)\). Clearly, \(e \equiv 0\) is a solution of equation (5), and the system (4) acts as a predictor for (3) if the error dynamics (5) has \(e = 0\) as asymptotically stable equilibrium. In this paper, we derive a sufficient condition for the zero solution of (5) to be asymptotically stable. We define the Jacobian of \(\phi(x, e)\) with respect to \(e\) at \(e = 0\) by

\[
D(x) \triangleq \left( \frac{\partial \phi(x, e)}{\partial e} \right)_{e=0}
\]

whose components are functions of \(x\). Then the linearization of (5) about \(e = 0\) is given by

\[
\begin{align*}
\dot{e} &= A_0 e + A_1 e_{\tau} + D(x)e.
\end{align*}
\]
It is well-known that if \( e = 0 \) of (6) is asymptotically stable, then \( e = 0 \) of (5) is also asymptotically stable (Kolmanovskii and Myshkis, 1999). Therefore, using this linearized system, we obtain the following lemma concerning the stability of the trivial solution \( e = 0 \) of (5).

**Lemma 1.** (Oguchi and Nijmeijer, 2004)
Assume the non-empty set \( \Omega \subset \mathbb{R}^n \) is such that all trajectories of the system \( \Sigma_A \) converge to it. If there exist symmetric and positive-definite matrices \( P, \Gamma_0, \Gamma_1 \) and \( \Gamma_2 \) of dimension \( n \times n \) satisfying

\[
\begin{align*}
(A_0 + A_d + D(x))T P + P(A_0 + A_d + D(x)) \\
+ \tau(A_d^T \Gamma_0 A_0 + A_d^T \Gamma_1 A_d + D(x)^T \Gamma_2 D(x)) \\
+ \tau P A_d (\Gamma_0^{-1} + \Gamma_1^{-1} + \Gamma_2^{-1}) A_d^T P < 0
\end{align*}
\]

for all \( x \in \Omega \), then the zero solution of (5) is asymptotically stable.

Using Lemma 1 we obtain the following theorem concerning how to choose the gain \( K \) stabilizing system (5).

**Theorem 2.** (Oguchi and Nijmeijer, 2004)
Let \( \tau > 0 \). If there exist a symmetric and positive-definite matrix \( P > 0 \) and a matrix \( Y \) such that the following holds for all \( x \in \Omega \):

\[
\begin{bmatrix}
M_{11} + M_{11}^T & M_{12} & M_{13} & M_{14} & M_{15} \\
M_{12}^T & -\frac{1}{2}P & 0 & 0 & 0 \\
M_{13} & 0 & -\frac{1}{2}P & 0 & 0 \\
M_{14} & 0 & 0 & -\frac{1}{2}P & 0 \\
M_{15} & 0 & 0 & 0 & -\frac{1}{2}P
\end{bmatrix} < 0
\]

where \( M_{11} = PA(x) + Y C, M_{12} = A_d^T P, M_{13} = A_d^T P + C^T Y T, M_{14} = PA_1 + Y C \) and \( M_{15} = D(x)^T P \); then the zero solution of system (5) is asymptotically stable and \( K = P^{-1} Y \) is a stabilizing gain matrix.

### 3 Robustness on perturbations

In the previous section, we assumed that the model of the master system is exactly known and an identical copy of the system can be constructed as the slave system. In a practical situation, however, systems to be coupled may have some mismatch and uncertainties of the parameters. Therefore in this paper, we consider to derive a condition for anticipating synchronization of systems with model uncertainties and perturbation.

We begin with the following general form of system with uncertainties and/or perturbation.

\[
\dot{x}(t) = f(t, x_0) + g(t, x_0)
\]

where \( x \in \mathbb{R}^n \), \( x_0 = x(t + \theta) \) for \( \theta \in [-\tau, 0] \) and \( g(t, x_0) \) denotes uncertainty and/or perturbation. Suppose now that the nominal system

\[
\dot{x}(t) = f(t, x_0)
\]

has an asymptotically stable equilibrium point at the origin. Then we obtain the following theorem.

**Theorem 3.** Let \( x = 0 \) be an asymptotically stable equilibrium point of the system (8). Let \( V \) be a Razumikhin function of the system (8) satisfying the following conditions 1, 2 and 3 for some positive constants \( c_i \) for \( i = 1, \ldots, 4 \). In addition, suppose the perturbation term \( g(t, x_0) \) satisfies the following conditions 4, 5 and 6 for some positive constants \( c_4 \) and \( \gamma \). Then, the origin is an asymptotically stable equilibrium point of the perturbed system (7).

1. \( c_1 \| x \|^2 \leq V(t, x) \leq c_2 \| x \|^2, \ t \geq t_0 - \tau, x \in B_H = \{x \in \mathbb{R}^n ||x|| < H \}
2. there exists a positive \( p > 1 \) such that

\[
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x_0) \leq -c_3 \| x \|^2
\]

for all \( t \geq t_0 \) and those continuous functions \( x(t + \theta) \) for \(-\tau \leq \theta \leq 0 \) satisfying

\[
V(t + \theta, x(t + \theta)) < pV(t, x(t))
\]

Moreover, if all the assumptions hold globally, then the origin is globally asymptotically stable.

**Proof.** From condition 4, \( x = 0 \) is an equilibrium point of the perturbed system (7). The existence of \( V \) satisfying condition 1 and 2 guarantees that the zero solution of the system (8) is asymptotically stable from the Lyapunov-Razumikhin theorem (Hale and Verduin Lunel, 1993; Kolmanovskii and Myshkis, 1999). The derivative of \( V \) along the trajectories of (7) is given by

\[
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x_0) + \frac{\partial V}{\partial x} g(t, x_0).
\]

If \( V(t + \theta, x(t + \theta)) < pV(t, x(t)), \ \theta \in [-\tau, 0] \) for \( t \geq t_0, \) then

\[
c_1 \| x(t + \theta) \|^2 \leq V(t + \theta, x(t + \theta)) < pV(t, x(t)) \leq pc_2 \| x(t) \|^2.
\]

and we obtain

\[
\| x(t + \theta) \| \leq \sqrt{\frac{pc_2}{c_1}} \| x(t) \|.
\]
Using condition 4,5 and 6, we obtain
\[
\dot{V}(t, x) \leq -c_3\|x\|^2 + \|\frac{\partial V}{\partial x}g(t, x_\theta)\|
\]
\[
\leq -c_3\|x\|^2 + c_4\|x\| \cdot \gamma \sup_{\theta \in [-\tau, 0]} \|x(t + \theta)\|.
\]
Substituting (9) into the above inequality yields
\[
\dot{V}(t, x) \leq -c_3\|x\|^2 + c_4\gamma \sqrt{\frac{p\varepsilon_2}{c_1}} \|x\|^2
\]
\[
\leq -\left(c_3 - c_4\gamma \sqrt{\frac{p\varepsilon_2}{c_1}}\right)\|x\|^2 < 0.
\]
From the Lyapunov-Razumikhin theorem, the zero solution of the system (7) is also asymptotically stable. □

As an example of Theorem 3, consider the following system.
\[
\dot{x}(t) = -ax(t - \ell)(1 + x(t)) + d(x_\theta)
\] 
(10)
where \(x \in \mathbb{R}\) and \(d\) denotes a perturbation satisfying \(d(0) = 0\) and \(\|d(x_\theta)\| \leq \gamma \sup_{\theta \in [-\tau, 0]} \|x(t + \theta)\|\). Since \(x(t)\) is continuously differentiable for \(t \geq 0\) one can write
\[
x(t - \ell) = x(t) - \int_{t-\ell}^{t} \dot{x}(s)ds = x(t) + a \int_{t-\ell}^{t} x(s - \ell)(1 + x(s))ds
\]
for \(t \geq \ell\). By using this expression for \(x(t - \ell)\) in (10) without perturbation \(d\), we obtain
\[
\dot{x}(t) = -ax(t) - a^2 \int_{t-\ell}^{t} x(s - \ell)ds
\]
\[
- a^2 \int_{t-\ell}^{t} x(s - \ell)x(s)ds - ax(t - \ell)x(t)
\]
for arbitrary continuous initial data on \([-2\ell, 0]\). If \(V(x) = x^2/2\), then, for any constant \(p = q^2 > 1\),
\[
\dot{V} \leq -a(1 - a\ell q)x^2(t) + aq(a\ell q + 1)x(t)^3
\]
\[
\leq -a\{1 - a\ell q - q(a\ell q + 1)H\}x^2(t)
\]
if \(V(x(t + \theta)) < pV(t)\) for \(-2\ell \leq \theta \leq 0\). If there exist some \(q > 1\) and \(H > 0\) satisfying \(c_3 = a\{1 - a\ell q - q(a\ell q + 1)H\} > 0\), the zero solution of (10) without \(d\) is uniformly asymptotically stable. Corresponding to Theorem 3, the perturbation term \(g(t, x_\theta)\) is given by
\[
g(t, x_\theta) = -a^2 \int_{t-\ell}^{t} d(x_\theta)ds + d(x_\theta)
\]
in the transformed system. Since \(g(t, 0) = 0\) and \(\|g(t, x_\theta)\| \leq \gamma |x(t + \theta)|\), by choosing \(c_1 = c_2 = 1/2\) and \(c_4 = 1\), we can conclude that the zero solution of the perturbed system (10) is also uniformly asymptotically stable under a perturbation satisfying \(\gamma < c_3/(a^2\ell q + q)\). Figure 2 shows a simulation result with \(a = 1\), \(\ell = 0.1\), \(p = 1.01\), \(d(x_\theta) = 0.4\sin(x(t))\) and \(\gamma = 0.4\).

4  Predictor-based Control
4.1 Configuration
In this section, as a practical application of the synchronization-based predictor, we consider a predictor based control of nonlinear systems with time-delay at the input. We consider the following system
\[
\dot{x} = f(x(t), u(t - \tau))
\] 
(11)
where \(f\) is smooth with respect to \(x\) and \(u\). We assume that when \(\tau = 0\) holds, i.e. the system is delay-free, a feedback \(u = \gamma(x(t))\) is so designed that the closed-loop system achieves satisfactory stability. However if \(\tau \neq 0\), the closed-loop system becomes \(\dot{x}(t) = f(x(t), \gamma(x(t - \tau)))\) and may be destabilized by the existence of the delay. While if we could obtain the future value \(x(t + \tau)\) of \(x\), by applying a feedback \(u(t) = \gamma(x(t + \tau))\), the closed-loop system would be given by \(\dot{x}(t) = f(x(t), \gamma(x(t)))\). From such a viewpoint, we propose a control scheme in which a predictor having the structure described in Section 2 is used to estimate the future value of a given system and the output of the predictor is used in place of the actual state of the system in a feedback. The synchronization-based predictor is described by
\[
\dot{z}(t) = f(z(t), u(t)) + KC[z(t - \tau) - x(t)]
\]
and the controller is given by \(u(t) = \gamma(z(t))\).

The configuration of the proposed control scheme is shown in Figure 3.

4.2 Separation Principle
We consider the stability of the total system. For system (11), a predictor and a control law are given by
\[
\dot{z}(t) = f(z(t), u(t)) + KC[z(t - \tau) - x(t)]
\] 
(12)
\[
u(t) = \gamma(z(t)).
\] 
(13)
Substituting (13) into (11) yields

\[ \dot{x}(t) = f(x(t), \gamma(z_r)) = f(x, \gamma(x + e)) \] (14)

where \( z_r \triangleq z(t - \tau) \) and \( e(t) \triangleq z(t - \tau) - x(t) \). The dynamics of the prediction error \( e(t) \) is given by

\[ \dot{e}(t) = f(z_r, \gamma(z_r)) - f(x, \gamma(z_r)) + KCe(t - \tau) \]

\[ = f(e + x, \gamma(z_r)) - f(x, \gamma(z_r)) + KCe_r \] (15)

If \( f \) and \( \gamma \) are sufficiently smooth to permit Taylor series expansion, we obtain

\[ f(x, \gamma(x + e)) = f(x, \gamma(x)) + \left( \frac{\partial f}{\partial y} \right) \left( \frac{\partial \gamma}{\partial x} \right) e + \mathcal{O}(e^2) \] (16)

\[ f(x + e, \gamma(z_r)) = f(x, \gamma(z_r)) + \left( \frac{\partial f}{\partial x} \right) e + \mathcal{O}(e^2) \] (17)

where \( \mathcal{O}(e^2) \) represents terms that converge to zero as \( \|e\|^2 \rightarrow 0 \). Substituting (16) and (17) into (14) and (15) respectively, we obtain

\[ \dot{x} = f(x, \gamma(x)) + \left( \frac{\partial f}{\partial x} \right) (\gamma(x)) e + \mathcal{O}(e^2) \]

\[ \dot{e} = \left( \frac{\partial f}{\partial x} \right) e + KCe_r + \mathcal{O}(e^2). \] (18)

As a result, when the prediction error converges to zero, the effect of the prediction error in equation (18) vanishes and then the behaviour of \( x \) coincides with \( \dot{x}(t) = f(x, \gamma(x)) \).

If the system is linear, this result corresponds to a counterpart of so-called the “separation principle” of controller and observer.

5 An Example

We consider a simplified boiler subsystem described by the following equation (Oguchi et al., 1994).

\[
\begin{align*}
\dot{x}(t) &= A_0x(t) + f(x(t)) \\
&\quad + g_1u_1(t - \ell_1) + g_2u_2(t - \ell_2) \\
y(t) &= \begin{bmatrix} h_1(x(t)) \\ h_2(x(t)) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_5(t) \end{bmatrix}
\end{align*}
\] (19)

where \( x \in \mathbb{R}^3, u_i \in \mathbb{R}, y \in \mathbb{R}^2 \),

\[
A_0 = \begin{bmatrix}
\frac{1}{T_1} & 0 & 0 & 0 & 0 \\
\frac{1}{T_2} & -\frac{1}{T_5} & -\frac{1}{T_4} & 0 & 0 \\
0 & 0 & -\frac{1}{T_4} & 0 & 0 \\
0 & -\frac{1}{T_5} & 0 & -\frac{1}{T_4} & 0 \\
0 & -\frac{1}{T_4} & 0 & -\frac{1}{T_4} & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
f(x) = \begin{bmatrix}
0, 1, x_3x_1(t), 0, 0, 0 \\
0, 0, 0, 0, 0
\end{bmatrix}^T,
g_1 = \begin{bmatrix}
\frac{1}{T_1} & 0, 0, 0, 0, 0
\end{bmatrix}^T, g_2 = \begin{bmatrix}
0, 0, 0, 0, 0
\end{bmatrix}^T
\]

where \( T_i, i = 1, \ldots, 5 \) and \( a_j, j = 1, 2, 3 \) are constants, and \( \ell_1 \) and \( \ell_2 \) are constant time-delays. Furthermore, the state variables \( x_i \), for \( i = 1, \ldots, 5 \) are normalized with percentage. Since \( x_5 \) is shifted the operating point \( \gamma \) into the zero, the bound of \( x_5 \) is given by \( -\gamma \leq x_5 \leq 100-\gamma \). Since this system has time-delays at the inputs, it is called an input time-delay system.

If \( \ell_1 = \ell_2 = 0 \) hold, then this system can be exactly linearized (Nijmeijer and van der Schaf, 1990; Isidori, 1995) by the following coordinate transformation and feedback.

\[
\xi(t) = \begin{bmatrix}
h_1(x), L_fh_1(x), h_2(x), L_fh_2(x), L^2_fh_2(x)
\end{bmatrix}^T
\]

\[
\begin{bmatrix}
u_1 \\ u_2
\end{bmatrix} = G(x)^{-1} \begin{bmatrix}
-L_h^2h_1(x) + v_1(t) \\ -L^2_fh_2(x) + v_2(t)
\end{bmatrix}
\] (20)

where the nonsingular matrix \( G(x) \) is given by

\[
G(x) = \begin{bmatrix}
L_{g1}L_fh_1(x) & L_{g2}L_fh_1(x) \\
L_{g1}L^2_fh_2(x) & L_{g2}L^2_fh_2(x)
\end{bmatrix}
\]

However, when \( \ell_i \) are not 0, all nonlinear terms cannot be cancelled out by applying the same feedback (20). In order to accomplish exact linearization for non-zero \( \ell_i \), the nonlinear terms at time \( t + \ell_i \) must be evaluated at time \( t \). Therefore we attempt to estimate the future value of the state \( x \) by using a predictor based on the synchronization discussed above.

Now we assume that \( \ell_1 = \ell_2 = \ell \). If the inputs have time-delays of different lengths, that is \( \ell_1 \neq \ell_2 \), then we can make the time-delays of all inputs equal to each other by adding time-delays of appropriate lengths to

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shorter delays. Then the state predictor based on synchronization is given by

\[
\dot{z}(t) = A_0 z(t) + f(z) + g_1 u_1(t) + g_2 u_2(t) + K \{ C z(t - \ell) - y(t) \} \quad (21)
\]

where \( z(\theta) = z_0 \) for \(-\ell \leq \theta \leq 0 \). The dynamics of the prediction error \( e(t) = z(t - \ell) - x(t) \) is given by

\[
\dot{e}(t) = A_0 e(t) + K C e(t - \ell) + \{ f(e + x) - f(x) \}
\]

\[
\dot{e}(t) \triangleq A_0 e(t) + K C e(t - \ell) + F(e, x) \quad (22)
\]

where \( F(e, x) \triangleq f(e + x) - f(x) \). The coupling gain \( K \) is designed as follows.

If there exist a symmetric and positive-definite matrix \( P > 0 \) and a matrix \( Y \) such that the following LMI holds:

\[
\begin{bmatrix}
M_{11} + M_{11}^T & A_0^T P C^T Y^T & Y C \\
PA_0 & -\frac{1}{2} P & 0 & 0 \\
Y C & 0 & -\frac{1}{2} P & 0 \\
C^T Y^T & 0 & 0 & -\frac{1}{2} P
\end{bmatrix} < 0 \quad (23)
\]

where \( M_{11} = PA_0 + YC \), then the zero solution of system:

\[
\dot{e}(t) = A_0 e(t) + K C e(t - \ell) \quad (24)
\]

is asymptotically stable and \( K = P^{-1} Y \) is a stabilizing gain matrix.

The obtained gain \( K \) guarantees only that the zero solution of (24) is asymptotically stable. To ensure that \( e \) of the error dynamics (22) converges to zero as \( t \to \infty \), we must consider the effect of \( F(e, x) \) on the stability of the zero solution. By using the fact

\[
e(t - \ell) = e(t) - \int_{t-\ell}^{0} \dot{e}(t + \theta) d\theta
\]

for \( t \geq \ell \), we can write the linearized system (24) as

\[
\dot{e}(t) = (A_0 + KC) e(t)
\]

\[
- \int_{-\ell}^{0} (KA_0 e(t + \theta) + (KC)^2 e(t + \theta)) d\theta
\]

(25)

with initial condition \( e(\theta) = z_0 - x(\theta) \) for \(-\ell \leq \theta \leq \ell \). From the fact that \( F(0, x) = 0 \), if the system (25) and \(-KC \int_{-\ell}^{0} F(e, x) ds + F(e, x) \) satisfy the conditions of Theorem 3, we can conclude that the zero solution of (22) is also asymptotically stable. Then it is guaranteed that the delayed output \( z(t - \ell) \) of the predictor (21) converges to \( x(t) \).

By combining this state predictor with the linearizing feedback (20), we can obtain the linearizing feedback with the state prediction as follows.

\[
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix} = G(z)^{-1} \begin{pmatrix}-L_2^T h_1(z) + v_1(t) \\
-L_2^T h_2(z) + v_2(t)
\end{pmatrix}
\]

\[
v_1 \triangleq -s_1 h_1(z) - s_2 L_f h_1(z) + y_{1,ref}(t)
\]

\[
v_2 \triangleq -s_2 h_2(z) - s_4 L_f h_2(z) - s_3 L_2^2 h_2(x) + y_{2,ref}(t)
\]

where \( s_i \) are coefficients of Hurwitz polynomials:

\[
\begin{align*}
\lambda^2 + s_2 \lambda + s_1 \\
\lambda^3 + s_3 \lambda^2 + s_4 \lambda + s_3
\end{align*}
\]

Figure 4 and 5 show a simulation result. In this simulation, the time-delay \( \ell = 2.0 \), the initial condition is given by \( y_1(t) = 20 \) and \( y_2(t) = 0 \) for \( t \leq 0 \), the stabilizing gain matrix by

\[
K = \begin{pmatrix}
-0.0518 & -0.0865 & 0 & 0.0019 \\
0.0001 & 0.0019 & 0 & -0.0005 & -0.1736
\end{pmatrix}^T
\]

Figure 4. Outputs \( y_1 \) and \( y_2 \)

Figure 5. Prediction error \( e(t) = z(t - \ell) - x(t) \)
and the coefficients $s_i$ are given by

$$(s_1, s_2, s_3, s_4, s_5) = (1.0, 2.0, 1.0, 3.0, 3.0).$$

In Figure 4, the reference signal $y_{1,\text{ref}}$ is drawn by the hashed line and $y_{2,\text{ref}}(t) = 0$ for $t \geq 0$. This figure shows that the boiler system is exactly linearized by applying the obtained feedback. The prediction error $e$ of the predictor (21) for $t \geq \ell$ is illustrated in Figure 5. This figure shows that the prediction error converges to zero and the predictor based on anticipating synchronization works well as the predictor of $x(t)$.

6 Conclusion

In this paper, we have derived a sufficient condition for robust convergence of time-delay nonlinear systems by using Lyapunov–Razumikhin approach. In addition, the predictor-based controller is proposed for systems with time-delay at the input, and the effectiveness is shown by a numerical simulation of a boiler control system. In this paper we have considered the case in which $||g(t,0)|| = 0$ is known. In this case, the origin $x = 0$ is also the equilibrium point of the perturbed system. This requirement should be relaxed and it will be the subject of future research.

References


Oguchi, T., A. Watanabe and T. Nakamizo (2002). Input-output linearization of retarded non-linear systems by using an extension of Lie derivative. *Int. J. Control* 75(8), 582–590.


