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Analysing nonlinear systems with Higher Order Sinusoidal Input Describing Functions

1. Abstract

For high precision motion systems, modelling and control design specifically oriented at friction effects is instrumental. The Sinusoidal Input Describing Function theory represents a solid mathematical framework for analysing non-linear system behaviour. This theory however limits the description of the non-linear system behaviour to an approximated linear relation between sinusoidal excitation and sinusoidal response. An extension to Higher Order Describing Functions can be realised by calculating the corresponding Fourier coefficients. The resulting Higher Order Sinusoidal Input Describing Functions (HOSIDFs) relate the magnitude and phase of the higher harmonics of the periodic system response to the magnitude and phase of a sinusoidal excitation. This paper describes two techniques to measure HOSIDFs. The first technique is FFT based. The second technique is based on IQ (=in phase/quadrature phase) demodulation. In a case study both techniques are used to measure the changes in dynamics due to friction as function of drive level in an electric motor.

2. Introduction

In the analysis and synthesis of dynamic systems, linearity is often a prerequisite. As the required performance of mechanical systems increases, non-linear behaviour becomes of interest due to its adverse influence on system performance. In positioning systems for example friction can lead to limit cycling, which deteriorates positioning accuracy and increases wear and power consumption. Often, servo controllers are used to reduce the position errors caused by friction. The complexity of these controllers varies from a basic PID action to sophisticated model based compensation schemes, combining advanced friction models with digital signal processing [1], [2]. Increasing demands on positioning performance call for a steady advance in the synthesis techniques of controllers. The influences of non-linear system behavior have to be taken into account. It is evident that one cannot do without reliable data both for validation of the sophisticated models as well as input for state dependent control actions like gain scheduling as put into practice in these advanced controllers. This requires practical but reliable measurement techniques, which are not limited to linear system behavior. In this work we would like to extend the well-known procedures from frequency response analysis for linear systems, towards a class of non-linear dynamical systems, with harmonic responses where amplitude dependent behavior is obvious. Some approaches have been addressing the describing function analysis [3], [4]. Although this theoretical framework allows for higher order analysis it ignores the influence of higher order components and in this way produces a linearized version of the non-linear system. In this paper in section 3 the extention of the classic Describing Function to Higher Order Sinusoidal Input Describing Functions (HOSIDFs) is proposed for the class of non-linear systems with harmonic responses. In section 4 two measurement techniques to determine HOSIDFs are presented and tested in a case study described in section 5.
3. Higher Order Sinusoidal Input Describing Function

Consider a stable, non-linear time invariant system with an odd non-linearity. Let $u(t) = \hat{a} \sin(\omega_0 t)$ be the input signal. The steady state system response $y(t)$ is considered to be periodic with the fundamental frequency $\omega_0$ of the input signal $u(t)$. Signal $y(t)$ can be written as a summation of harmonics of the input signal $u(t)$, each with an amplitude and phase, which can depend on the amplitude $\hat{a}$ and frequency $\omega$ of the input signal. The describing function $H(\hat{a}, \omega)$ of the system is the complex ratio of the fundamental component $\tilde{y}(t)$ of the system response and the input sinusoid $u(t)$, see figure 1.

$$u(t) = \hat{a} \sin(\omega_0 t) \quad \Rightarrow \quad \tilde{y}(t) = A_1(\hat{a}, \omega) \sin(\omega_0 t + \varphi_1(\hat{a}, \omega))$$

![Figure 1: Describing function representation](image)

The describing function $H(\hat{a}, \omega)$ is defined as [5]:

$$H(\hat{a}, \omega) = \frac{A_1(\hat{a}, \omega)e^{j(\omega_0 t + \varphi(\hat{a}, \omega))}}{\hat{a}e^{j\omega_0 t}} = \frac{1}{\hat{a}}(b_1 + ja_1) \quad (1)$$

The Fourier coefficients $a_1$ and $b_1$ are calculated as in (2), (3) with $T_0 = \frac{1}{\omega_0}$.

$$a_1 = \frac{2}{T_0} \int_{t_0}^{t_0 + T_0} y(t) \cos(\omega_0 t) \, dt \quad (2)$$

$$b_1 = \frac{2}{T_0} \int_{t_0}^{t_0 + T_0} y(t) \sin(\omega_0 t) \, dt \quad (3)$$

The describing function as defined above can be interpreted as the first order representation of a more global describing function $H_n(\hat{a}, \omega)$, see figure 2. This function can be defined as the complex ratio of the $n^{th}$ harmonic component in the output signal to a virtual $n^{th}$ harmonic signal derived from the excitation signal, (4). This virtual harmonic has equal amplitude as the fundamental sinusoid and zero phase.

$$H_n(\hat{a}, \omega) = \frac{A_n(\hat{a}, \omega)e^{j(n\omega_0 t + \varphi_n(\hat{a}, \omega))}}{\hat{a}e^{j\omega_0 t}} = \frac{A_n(\hat{a}, \omega)e^{j\varphi_n(\hat{a}, \omega)}}{\hat{a}} = \frac{1}{\hat{a}}(b_n + ja_n) \quad (4)$$

In this paper $H_n(\hat{a}, \omega)$ will be referred to as the Higher Order Sinusoidal Input Describing Function (HOSIDF).

![Figure 2: Higher order sinusoidal input describing function representation](image)
4. Measurement techniques for HOSIDFs

FFT method

In this measurement technique both the input signal \( u(t) \) and output signal \( y(t) \) are Fourier transformed. The data block length is chosen equal to \( p \) times the period \( T_0 \) of the excitation signal. This assures that all the power of the excitation signal is concentrated in line \( p \). The power of the response signal is fully concentrated in frequency lines \( n*p \) so leakage is absent. Let us consider the calculation of the \( k \)th order HOSIDF, see figure 3.

\[
\mathcal{H}(\hat{a}, \omega) = \frac{1}{2} \left[ A_k(\hat{a}, \omega) \sin(k\omega_0 t + \varphi + \varphi_k(\hat{a}, \omega)) - A_k(\hat{a}, \omega) \cos(k\omega_0 t + \varphi + \varphi_k(\hat{a}, \omega)) \right]
\]

The frequency line \( p \) with value \( a_p + jb_p \) represents the input signal. The Fourier coefficients \( a_p \) and \( b_p \) are calculated as in (2), (3). The \( k \)th component of the output signal is contained in frequency line \( k*p \) of its spectrum and has the complex value \( a_{kp} + jb_{kp} \). The amplitudes \( \hat{a} \) of the excitation signal and \( A_k(\hat{a}, \omega) \) of the \( k \)th harmonic component in the output signal are calculated from the autospectra. The phase \( \varphi_k(\hat{a}, \omega) \) of the \( k \)th order describing function is the phase of the cross spectrum between the two FFT lines containing the excitation signal and its \( k \)th harmonic component of the output signal.

IQ demodulation method

An alternative to the FFT method is the IQ demodulation method [6], [7]. In this method \( n \) IQ demodulators decompose the system response signal into \( n \) components. In figure 4 the IQ demodulator for the \( k \)th harmonic of the output signal is explained in more detail. The signal is multiplied with \( \sin(k\omega_0) \) and \( \cos(k\omega_0) \) in two separate branches. These multiplications result in the generation of two new signals, each consisting of the sum and difference frequencies of the original signal and the oscillator signals. These new signals are 90 deg apart. After low-pass filtering the remaining signals representing the \( k \)th harmonic are \( A_k(\hat{a}, \omega) \sin(\varphi_k(\hat{a}, \omega)) \) called the I-signal (= in phase) component and \( A_k(\hat{a}, \omega) \cos(\varphi_k(\hat{a}, \omega)) \) called the Q signal (= quadrature) component. From the \( I_k \) and \( Q_k \) components \( A_k(\hat{a}, \omega) \) and \( \varphi_k(\hat{a}, \omega) \) are computed.
\[
\sum_{n=1}^{\infty} A_n(\dot{\omega}, \omega) \sin(n\omega_0 t + \varphi_n(\dot{\omega}, \omega))
\]

\[
2 \cos(k\omega_0 t)
\]

\[
2 \sin(k\omega_0 t)
\]

\[\text{Lowpass filter}\]

\[
I_k = A_k(\dot{\omega}, \omega) \sin(\varphi_k(\dot{\omega}, \omega))
\]

\[
A_k(\dot{\omega}, \omega) = \sqrt{I_k^2 + Q_k^2}
\]

\[
\varphi_k(\dot{\omega}, \omega) = \arctan\left(\frac{Q_k}{I_k}\right)
\]

\[
Q_k = A_k(\dot{\omega}, \omega) \cos(\varphi_k(\dot{\omega}, \omega))
\]

Figure 4: Determination of \(k\)th order HOSIDF using IQ demodulation

5. Case study

In this case study the HOSIDFs of a real mechanical system with friction are measured. The test object is a 20 W electric DC collector motor with encoder. The motor is powered by a voltage-to-current converter see figure 5. The input to the system, i.e. the motor current \(I_m\), is measured using a current probe with a sensitivity of 2 A/V. The response signal is angular velocity \(\omega_{out}\). For small rotations this angular velocity can be measured with a dual fibre laser vibrometer as the linear velocity difference between two points spaced 180 deg on the circumference of the shaft divided by the spacing of the points. The resulting sensitivity is 0.588 rad/s/V. A block diagram of the measurement set-up is given in figure 6. \(J_1\) and \(J_2\) represent the inertias of the motor and the encoder. \(T\) is the driving torque and \(C\) the stiffness of the motor shaft. As in the LuGre model [8] the friction influence is modelled with \(\sigma_0\) the bristle stiffness, \(\sigma_1\) the bristle damping and \(\sigma_2\) the viscous damping. The combination of the bristle stiffness \(\sigma_0\) and the inertias \(J_1\) and \(J_2\) will cause a friction-induced resonance which frequency will depend upon the excitation level [9], [10].

To investigate this non-linear behaviour, the HOSIDFs were determined using the measurement techniques described in 4. The frequency of the generator signal was chosen 320 Hz to excite the system both above and below its friction induced resonance frequency depending upon the instantaneous amplitude of the excitation signal. Other considerations were that 320 Hz is not a multiple of the 50 Hz mains frequency and that the signal can be generated with an integer number of 12.8 kHz samples per period, being one of the sampling frequencies of the SigLab 20-42 dynamic

Figure 5: Case study on a small DC motor

Figure 6: Block diagram of measurement set-up
signal analyser. The main parameters used for the FFT method are a block-size of 1600 samples and a sampling frequency of 12.8 kHz so $\Delta f = 8$ Hz, Hamming window, no overlap processing. The low-pass filters for the IQ method are 8th order Butterworth with 4 Hz cut-off frequencies. Figure 7 shows the amplitude dependency of the HOSIDFs measured at the fixed frequency of 320Hz. The solid line shows the results from the IQ method, the dots indicate the measurements from the block based FFT method. In the left column the magnitude plots are presented for the odd order Sinusoidal Input Describing Functions (SIDF). The even orders are all zero due to the odd characteristics of the non-linearity. The right hand column gives the corresponding phase relations. In the magnitude plot of the first order SIDF we can distinguish three regions. From 0 to approximately 0.5 mA the system gain is excitation independent. Between 0.5 mA and approximately 2.5 mA a strong excitation level dependency is visible. Above 2.5 mA the gain is independent of the excitation level at a stable 18 dB but the system remains non-linear as can be concluded from the plots of the higher order SIDFs. The gain of the third order FRF decreases initially until it reaches a minimum at an excitation of 0.5 mA. This is due to the low signal to noise ratios in this region resulting in large uncertainties in the calculations. For increasing excitation its magnitude increases and reaches a maximum of $-8$ dB at approximately 2.5 mA. Above that excitation level the gain decreases again slightly. The same pattern is visible for the fifth order SIDF, however its maximum of $-15$ dB is reached at an excitation level of 4.5 mA.

**Figure 7: HOSIDFs measured with IQ and FFT method**

6. Discussion

An important difference between the FFT method and the IQ method is the way of processing data. The FFT method is block oriented and generates results every 0.125 sec where the IQ method is sample based so the time between two measurements equals the sample period of 1/12800 sec. This results in less measurement points in the corresponding HOSIDF plots as can be seen in figure 7. Apart from this difference the results of the two methods are substantially equivalent. The visible differences which are mainly concentrated in the range of low excitation levels are probably due to low signal to noise ratios.
7. Conclusion and further research

An extension of the theory of Describing Functions was presented. Higher Order Sinusoidal Input Describing Functions (HOSIDF) can be defined as the excitation amplitude dependent gain and phase relations between a virtual set of harmonics of the excitation signal of a non-linear system and the corresponding real harmonics in the output signal. The theory developed assumes the non-linear system to respond harmonically at a sinusoidal excitation. To measure the HOSIDF two measurement methods were described. The first method is FFT based and uses auto and cross spectrum information to determine the system gain and phase of the HOSIDF as function of the input signal amplitude and frequency. The second measurement technique uses IQ demodulation techniques and is sample based. A case study was presented describing the application of the two measurement techniques. In this case study the device under test was a small current fed DC electric motor. Due to its construction this motor exhibits a significant amount of friction. With both measurement techniques the HOSIDFs have been successfully identified. The results clearly show non-linear system behaviour as function of excitation level.

Since the HOSIDF is not only a function of amplitude but also of excitation frequency many single frequency measurements have to be done in order to quantify its frequency dependency. Future work will comprise merging the multi-sine excitation techniques [11] with the HOSIDF technique described in this paper. Implementation of this combined knowledge in hardware like FPGAs will hopefully result in the construction of a new, valuable and practical measurement tool.

References:


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