Adaptive Mesh Refinement Techniques For Spectral Elements

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Introduction
Spectral element methods are weighted residual techniques for the numerical solution of partial differential equations. For the development of adaptive spectral element techniques we need two key ingredients: a non-conforming discretization and single mesh a posteriori error estimators [1,2].

Non-Conforming Discretization
The extension to the original spectral element method, the Mortar Element Method, introduces the constrained approximation idea. We minimize the difference in function values across each non-conforming interface (see Figure 1, right) to make the basis as continuous as possible. We enforce the weighted residual equation:

\[ \int_{\Gamma_i} (u - v) \psi \, ds = 0, \quad \forall \psi \in P_{N-2}(\Gamma_i) \]

where \( u \) and \( v \) are two functions that we would like to be continuous, and \( \psi \) is the weight to perform the minimization.

Error Estimators
The error estimates are single mesh a posteriori [2] local per element error estimates consisting of:

- the integration to infinity of the least squares best fit extrapolated Legendre coefficients, representing the norm of the error due to truncation, \( \| u - \Pi_h \tilde{u} \| \)
- the norms of the error due to approximating the exact coefficients numerically by quadrature, or \( \| u_h - \Pi_h \tilde{u} \| \)

Figure 1 (left) The approximation error contributions; (right) A non-conforming grid, \( v \) represents the mortar(non-conform edge) endpoints, \( \gamma_i \) are the mortars.

Implementation-2D
The mortar spectral element method has been implemented using a combination of FORTRAN, C and C++. Figure 2 illustrates the architecture of the implementation, where VDB is the voxel data base that keeps the geometric positions of the non-conforming mesh.

Results
We consider the unsteady rotation of a Gaussian hill described by the convection equation in two dimensions with domain \( \Omega = [-1, 1]^2 \):

\[ \partial c + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = 0 \]  

where \( u = -\omega y, v = \omega x, \omega = 2\pi \) (See Figure 3).

Figure 2 The architecture of the mortar implementation.

Adaptation based on the local Legendre spectrum. From left to write: step 1 spectral mesh; step 50; step 100; step 190.

The next example is a Gaussian 2-d steady distribution on a uniform grid. We compare the relative errors for solving equation for the conforming and non-conforming case (see Figure 4):

\[ \Delta u = (400^2 r^2 - 800)e^{-400r^2/2} \]  

where \( r^2 = x^2 + y^2 \), on \([-0.5, 0.5]^2\).

Figure 4 From left to write: adaption based on the local Legendre spectrum, tolerance=0.4E-09; adaption based on solution gradient, tolerance=0.0096; conform, errors versus no.of mesh-points; non-conform, error versus no. of mesh points.

Conclusions
We have outlined the basic of the implementation of an adaptive spectral method. The mortar discretizations represent a significant advance for spectral element methods. The combination of the nonconforming formulation with the error estimators represent the basis for a fully adaptive method.

References:

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