A Controlled Closing Theorem

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1. INTRODUCTION

In a number of problems in the theory of nonlinear oscillations and the theory of nonlinear control systems, one must investigate the behavior of solutions of differential systems under small changes of right-hand sides [1, p. 88; 2, p. 76; 3]. In particular, the following problem is of interest: is it possible to transform a nearly periodic motion (for example, an almost periodic or recurrent motion) into a periodic motion with the use of small admissible changes of the right-hand side? The positive answer to this question was given in [4, p. 80 of the Russian translation], where the so-called closing lemma was proved (see also [2, p. 76]). However, in problems of control of oscillations, which have been intensively studied in the recent years, arbitrary variations of the right-hand sides are not allowed; only variations compatible with the actual capabilities of the control are admissible. Therefore, it is of interest to generalize the closing lemma to controlled systems. Some conditions of this type are stated in [3, 5, p. 121]. They essentially pertain to control theory and provide a criterion for the controllability of a nonlinear system near a recurrent trajectory by controls small in the uniform metric.

In the present paper, we give a rigorous statement and a complete proof of this assertion (referred to as the controlled closing theorem).

2. THE CONTROLLED CLOSING THEOREM

Let us consider the nonlinear system

\[ \dot{x} = F(x, u), \]

where \( x \in \mathbb{R}^n \) is the system state vector, \( u : \mathbb{R}_+ \to \mathbb{R} \) is a scalar control function, and \( F : \mathbb{R}^{n+1} \to \mathbb{R}^n \) is a sufficiently smooth function. (Exact smoothness conditions will be imposed on \( F \) in the statement of the theorem.) Along with this system, we consider system (1) with \( u \equiv 0 \), that is,

\[ \dot{x} = f(x), \]

where \( f(x) = F(x, 0) \). By \( \bar{x}(t, x_0) \) we denote the trajectory of system (2) with the initial condition \( \bar{x}(0, x_0) = x_0 \).

Let us give some relevant definitions.

Definition 1. A point \( x_0 \in \mathbb{R}^n \) is called a strongly nonwandering point of the autonomous system (2) if for some number \( \tau_0 > 0 \) and for any \( \varepsilon > 0 \), there exists a \( \tau_\varepsilon > \tau_0 \) such that \( \| \bar{x}(\tau_\varepsilon, x_0) - x_0 \| < \varepsilon \).

Definition 2. A function \( \varphi : \mathbb{R}_+ \to \mathbb{R}^n \) is said to be recurrent if for some number \( \tau_0 > 0 \) and for any \( \varepsilon > 0 \), there exists a \( T_\varepsilon > \tau_0 \) such that the following property holds. For any \( t \geq 0 \), there exists a number \( \tau_\varepsilon, \tau_0 < \tau_\varepsilon < T_\varepsilon \), such that \( \| \varphi(t + \tau_\varepsilon) - \varphi(t) \| < \varepsilon \).

We introduce the following notation. For a function \( \Gamma \) with arguments \( \xi_1, \ldots, \xi_m \), we write \( \Gamma \in C^k_{\xi_1, \ldots, \xi_m} \) if \( \Gamma \) is \( C^k \) in the variable \( \xi_i, i = 1, \ldots, m \). If \( \Gamma \) depends only on \( \xi_1 \), then we write \( C^k_{\xi} \) instead of \( C^k_{\xi_1, \ldots, \xi_m} \). For a mapping \( \Gamma \), the symbols \( \partial \Gamma / \partial \xi \) and \( \Gamma'_\xi \) stand for the Jacobi matrix.
We set \( g(x) = \partial F(x,0)/\partial u \) and \( \text{ad}_f g(x) = [f'_x g - g'_x f](x) \) Let \( \Phi(t,x_0) \) be the principal solution matrix of the variational equation around \( \ddot{x}(t,x_0) \), i.e.,

\[
\dot{\Phi} = f'_x(\ddot{x}(t,x_0))\Phi, \quad \Phi(0,x_0) = I_n.
\] (3)

We set \( t_k = k\Delta, \Delta > 0 \), and define a function \( u(t) \) for \( t \in [0,t_n] \) as follows:

\[
u(t) = u_i, \quad t \in [t_i,t_{i+1}), \quad i = 0,\ldots,n-1,
\] (4)

where the \( u_i \) are the components of a vector \( u = [u_i]_{i=0}^{n-1} \in \mathbb{R}^n \). By \( x(t,u,x_0), t \in [0,t_n] \), we denote the solution of system (1) with the control (4) and the initial condition \( x(0,u,x_0) = x_0 \).

The following statement is the main result of the present paper.

**The Controlled Closing Theorem.** Suppose that the function \( F : \mathbb{R}^{n+1} \to \mathbb{R}^n \) occurring in system (1) and a point \( \ddot{x}_0 \in \mathbb{R}^n \) satisfy the following conditions:

1. \( \ddot{x}_0 \) is a strongly nonwandering point of system (2);
2. \( F \in C_{x,u}^{n,1} \);
3. \( \det\, [g, \text{ad}_f g, \ldots, \text{ad}_f^{n-1} g](\ddot{x}_0) \neq 0 \).

Then for any \( \varepsilon > 0 \), there exists a number \( \tau_\varepsilon > 0 \) and a step function \( u_\varepsilon : [0,\tau_\varepsilon) \to \mathbb{R} \) such that

\[
\sup_{t \in [0,\tau_\varepsilon)}|u_\varepsilon(t)| < \varepsilon,
\] (5)

\[
\dot{x} = F(x,u_\varepsilon(t)), \quad x(0) = \ddot{x}_0, \quad x(\tau_\varepsilon) = \ddot{x}_0.
\] (6)

**Corollary** (the closing theorem for a recurrent trajectory). The assertion of the theorem remains valid if condition (1) is replaced by the following requirement:

(1) \( \ddot{x}(t,\ddot{x}_0) \) is a recurrent trajectory of system (2).

**Proof of the corollary.** Condition (1a) implies condition (1). Indeed, if the trajectory \( \ddot{x}(t,\ddot{x}_0) \) is recurrent, then we readily see that \( \ddot{x}(t,\ddot{x}_0) \) is a strongly nonwandering point of system (2) for each \( t \geq 0 \). In particular, so is \( \ddot{x}_0 \).

Let us clarify the idea of the proof of the theorem. Since \( \ddot{x}_0 \) is strongly nonwandering, we choose a point \( \ddot{x} = \ddot{x}(\tau_\varepsilon,\ddot{x}_0) \) arbitrarily close to \( \ddot{x}_0 \). Then we move along the trajectory \( \ddot{x}(t,\ddot{x}_0) \) backward by the time \( n\Delta_0 \), where \( \Delta_0 > 0 \) is chosen in a special way, and join the points \( \ddot{x}(\tau_\varepsilon - n\Delta_0,\ddot{x}_0) \) and \( \ddot{x}_0 \) with the use of a piecewise constant control of the form (4). The existence of such a control is guaranteed by the implicit function theorem [in particular, by condition (3)], and the control is small since the points \( \ddot{x} \) and \( \ddot{x}_0 \) are close to each other. We define the desired control to be zero for \( t \in [0,\tau_\varepsilon - n\Delta_0] \) and to be equal to the above-mentioned piecewise constant control for \( t \in [\tau_\varepsilon - n\Delta_0,\tau_\varepsilon] \).

Prior to giving a rigorous proof of the theorem, we establish some auxiliary assertions.

**Lemma 1.** Let \( F \in C^1 \); then \( x(t_n,u,x_0) \in C_{x_0,u}^{n,1} \) and

\[
\partial x (t_n,0,x_0)/\partial u_l = \Phi(t_n,x_0) \int_{t_i}^{t_{i+1}} \Phi^{-1}(s,x_0) g(\ddot{x}(s,x_0)) \, ds, \quad l = 0,\ldots,n-1.
\] (7)

**Proof.** Let \( X(t,u,\tau,\chi_0) \) be the solution of the Cauchy problem

\[
\dot{X} = F(X,u), \quad X(\tau,u,\tau,\chi_0) = \chi_0,
\]

where \( u \) is a scalar constant treated as a parameter. It follows from the definition of \( \ddot{x}(t,x_0) \) that

\[
X(t,0,\tau,\ddot{x}(\tau,x_0)) = \ddot{x}(t,x_0).
\] (8)
Since \( F \in C^1 \), it follows from the theorem on the smooth dependence of the solution of the Cauchy problem on the parameter and the initial data \([6, p. 120]\) that \( X (t, u, \tau, \chi_0) \) is a function smooth with respect to all arguments, \( \partial X (t, u, \tau, \chi_0) / \partial u \) satisfies the Cauchy problem
\[
\dot{\psi} = (\partial F (X (t, u, \tau, \chi_0), u) / \partial x) \psi + \partial F (X (t, u, \tau, \chi_0), u) / \partial u, \quad \psi(\tau) = 0,
\]
and \( \partial X (t, u, \tau, \chi_0) / \partial \chi_0 \) satisfies the problem
\[
\dot{\varphi} = (\partial F (X (t, u, \tau, \chi_0), u) / \partial x) \varphi, \quad \varphi(\tau) = I_n.
\]
With regard for (8) and (3), we obtain
\[
\partial X (t, 0, \tau, x (\tau, x_0)) / \partial \chi_0 = \Phi (t, x_0) \Phi^{-1} (\tau, x_0).
\]
Using the Cauchy formula for the solution of a nonhomogeneous linear system of differential equations, we obtain
\[
\frac{\partial X}{\partial u} (t, 0, \tau, x (\tau, x_0)) = \Phi (t, x_0) \int_0^t \Phi^{-1} (s, x_0) g \left( x \left( s, x_0 \right) \right) ds.
\]
We prove the lemma by induction over the index \( i \) of the point \( t_i \) at which the derivative is evaluated.
Let \( i = 1 \). Then we have \( x (t, u, x_0) = X (t, u_0, 0, x_0), \ t \in [0, t_1]. \) As was proved above, \( X (t_1, u_0, 0, x_0) \in C^{1,1}_{x_0, u_0}. \) Consequently, since \( x (t_1, u, x_0) \) is independent of \( u_1, \ldots, u_{i-1}, \) it follows that \( x (t_1, u, x_0) \in C^{1,1}_{x_0, u} \) and \( \partial x (t_1, 0, x_0) / \partial u_l = 0, \ l = 1, \ldots, n - 1. \) By (10), we have
\[
\frac{\partial x}{\partial u_0} (t_1, 0, x_0) = \Phi (t_1, x_0) \int_0^{t_1} \Phi^{-1} (s, x_0) g \left( x \left( s, x_0 \right) \right) ds.
\]
Let \( x (t_i, u, x_0) \in C^{1,1}_{x_0, u} \) and
\[
\frac{\partial x}{\partial u_l} (t_i, 0, x_0) = \Phi (t_i, x_0) \int_{t_i}^{t_{i+1}} \Phi^{-1} (s, x_0) g \left( x \left( s, x_0 \right) \right) ds, \quad l = 0, \ldots, i - 1, \tag{11}
\]
for some \( i < n. \)
Let us show that this assertion remains valid for \( i + 1. \) Since \( x (t, u, x_0) = X (t, u_i, t_i, x (t_i, u, x_0)), \ t \in [t_i, t_{i+1}], \) it follows that \( x (t, u, x_0) \) is a composition of two smooth functions, and hence \( x (t_{i+1}, u, x_0) \in C^{1,1}_{x_0, u}. \) By (10), we have
\[
\frac{\partial x}{\partial u_l} (t_{i+1}, 0, x_0) = \partial X (t_{i+1}, 0, t_i, x (t_i, x_0)) = \Phi (t_{i+1}, x_0) \int_{t_i}^{t_{i+1}} \Phi^{-1} (s, x_0) g \left( x \left( s, x_0 \right) \right) ds.
\]
Relations (9) and (11) imply that
\[
\frac{\partial x}{\partial u_l} (t_{i+1}, 0, x_0) = \frac{\partial X}{\partial x_0} (t_{i+1}, 0, t_i, x (t_i, x_0)) \frac{\partial x}{\partial u_l} (t_i, 0, x_0) = \Phi (t_{i+1}, x_0) \int_{t_i}^{t_{i+1}} \Phi^{-1} (s, x_0) g \left( x \left( s, x_0 \right) \right) ds, \quad l = 0, \ldots, i - 1.
\]
Since \( x(t_{i+1}, u, x_0) \) is independent of \( u_{i+1}, \ldots, u_{n-1} \), we have \( \partial x (t_{i+1}, 0, x_0) / \partial u_k = 0 \), \( k = i + 1, \ldots, n - 1 \), which completes the induction step. We have thereby proved formula (11) for \( i = 1, \ldots, n \), and for \( i = n \), we arrive at the assertion of the lemma.

Let us construct a mapping \( \mathcal{F}_\Delta : \mathbb{R}^n \to \mathbb{R}^n \) as follows:

\[
\mathcal{F}_\Delta (x, u) = x (t_n, u, \bar{x} (-t_n, x)) - \bar{x}_0,
\]

where \( \Delta > 0 \) is the parameter occurring in the definition of \( t_n \).

**Lemma 2.** Let \( F \in C^{n,1}_{x,u} \) and

\[
det [g, \text{ad}_f g, \ldots, \text{ad}_f^{n-1} g] (\bar{x}_0) \neq 0.
\]

Then there exists a number \( a > 0 \) such that \( \det [\partial \mathcal{F}_\Delta (\bar{x}_0, 0) / \partial u] \neq 0 \) for \( 0 < \Delta \leq a \).

**Proof.** Definition (12) of the mapping \( \mathcal{F}_\Delta \) implies that

\[
\partial \mathcal{F}_\Delta (\bar{x}_0, 0) / \partial u = \partial x (t_n, 0, \bar{x} (-t_n, \bar{x}_0)) / \partial u.
\]

We expand this into a series, thus obtaining

\[
d (\Phi^{-1} (s, x_0) g (\bar{x} (s, x_0))) / ds = -\Phi^{-1} (s, x_0) [f'_s g - g'_s f] (\bar{x} (s, x_0))
\]

\[
= -\Phi^{-1} (s, x_0) \text{ad}_f g (\bar{x} (s, x_0)).
\]

By induction, we have

\[
d^k (\Phi^{-1} (s, x_0) g (\bar{x} (s, x_0))) / ds^k = (-1)^k \Phi^{-1} (s, x_0) \text{ad}_f^k g (\bar{x} (s, x_0)), \quad k = 0, \ldots, n.
\]

We expand this into a series, thus obtaining

\[
\Phi^{-1} (s, x_0) g (\bar{x} (s, x_0)) = \Phi^{-1} (t_n, x_0) \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \text{ad}_f^k g (\bar{x} (t_n, x_0)) (s - t_n)^k + \xi (s, x_0) (s - t_n)^n,
\]

where \( \xi (s, x_0) = ((-1)^n / n!) \Phi^{-1} (\tilde{s}, x_0) g (\bar{x} (\tilde{s}, x_0)) \) and \( \tilde{s} \) is some point in the interval \( (s, t_n) \).

Let us substitute the resulting expression into (7):

\[
\partial x (t_n, 0, x_0) / \partial u_l = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \text{ad}_f^k g (\bar{x} (t_n, x_0)) \int_{t_l}^{t_{l+1}} (s - t_n)^k ds + \xi_l (\Delta, x_0)
\]

\[
= \sum_{k=1}^{n} \frac{\Delta^k}{k!} \text{ad}_f^{k-1} g (\bar{x} (t_n, x_0)) [(n - l)^k - (n - l - 1)^k] + \xi_l (\Delta, x_0),
\]

where \( \xi_l (\Delta, x_0) = \Phi (t_n, x_0) f_l^{t_{l+1}} \xi (s, x_0) (s - t_n)^n ds. \)

Note that in the last passage, we have used the relation \( t_k = k \Delta. \) Replacing \( x (-t_n, \bar{x}_0) \) by \( x_0 \) in (16), we obtain, with regard to (14),

\[
\frac{\partial \mathcal{F}_\Delta}{\partial u_l} (\bar{x}_0, 0) = \sum_{k=1}^{n} \frac{\Delta^k}{k!} \text{ad}_f^{k-1} g (\bar{x}_0) [(n - l)^k - (n - l - 1)^k] + \tilde{\xi}_l (\Delta),
\]

where \( \tilde{\xi}_l (\Delta) = \xi_l (\Delta, \bar{x} (-t_n, \bar{x}_0)). \)
We set \( \chi(\Delta) = \bar{x}(-t_n, \bar{x}_0) \). Then in the explicit form, we have

\[
\xi_l(\Delta) = \Phi \left( t_n, \chi(\Delta) \right) \left( \frac{(-1)^n}{n!} \int_{t_i}^{t_{i+1}} \Phi^{-1}(\bar{s}, \chi(\Delta)) \text{ad}_j^l g(\bar{s}, \chi(\Delta)) (s-t_n)^n ds, \right)
\]

where \( \bar{s} \in (s, t_n) \).

The function \( \chi(\Delta) = \bar{x}(-\Delta n, \bar{x}_0) \) is continuous, since it is a solution of a differential equation, and \( \text{ad}_j^l g(x) \) is continuous by virtue of the smoothness requirements imposed on \( F \) in the lemma.

Since \( \Phi^{-1}(s, x) \) and \( \Phi(s, x) \) are solutions of the Cauchy problems (3) and (15) with right-hand sides smoothly depending on \( s \) and \( x \), it follows from the theorem on the differentiability of solutions with respect to the initial data and parameters [6, p. 120] that \( \Phi^{-1}(s, x) \in C^{1,1}_{s, x_0} \) and \( \Phi(s, x) \in C^{1,1}_{s, x_0} \). Consequently, \( \Phi^{-1}(s, \chi(\Delta)) \), \( \text{ad}_j^l g(\bar{s}, \chi(\Delta)) \), and \( \Phi(t_n, \chi(\Delta)) \) are continuous in \( s \) and \( \Delta \) and hence are bounded on the set \( \Delta \in [0, a_1] \), \( s \in [0, \Delta n] \), where \( a_1 > 0 \) is an arbitrary number. Hence we obtain the estimate

\[
\|\xi_l(\Delta)\| \leq C \Delta^{-n+1}, \quad l = 0, \ldots, n-1, \quad \Delta \in [0, a_1].
\]  

We set

\[
\psi_l(\Delta) = \sum_{i=1}^{n-1} \frac{\partial \varphi}{\partial u_i}(\bar{x}_0, 0), \quad l = 0, \ldots, n-1.
\]  

This, together with (17), implies that

\[
\psi_l(\Delta) = \sum_{k=1}^{n} \frac{\Delta^n}{k!} \text{ad}_j^{k-1} g(\bar{x}_0) (n-l)^k + \sum_{i=1}^{n-1} \xi_l(\Delta).
\]  

We set

\[
\Psi(\Delta) = [\psi_l(\Delta)]_{l=0}^{n-1}, \quad A = \left[ \frac{1}{k!} \text{ad}_j^{k-1} g(\bar{x}_0) \right]_{k=1}^{n}, \quad \Xi(\Delta) = \left[ \sum_{i=1}^{n-1} \xi_l(\Delta) \right]_{l=0}^{n-1}, \quad V = ((n-j+1)^j)_{i,j=1}^{n}, \quad D(\Delta) = \text{diag} [\Delta, \ldots, \Delta^n].
\]  

It follows from the estimate (18) and the definition of \( \Xi(\Delta) \) that

\[
\|\Xi(\Delta)\| \leq C_1 \Delta^{-n+1} \quad \text{for} \quad \Delta \in [0, a_1].
\]  

We rewrite relation (20) in the new notation: \( \Psi(\Delta) = AVD(\Delta) + \Xi(\Delta) \). By (13), \( \det A \neq 0 \); furthermore, \( \det V \neq 0 \) since this is a Vandermonde determinant, and \( \det D(\Delta) = \Delta^{n(n+1)/2} \neq 0 \) for \( \Delta \neq 0 \). Let \( \Delta > 0 \). Consider \( \det ([AVD(\Delta)]^{-1}\Psi(\Delta)) \) = \( \det [I_n + (AVD(\Delta))^{-1}\Xi(\Delta)] \). From the form of the matrix \( D(\Delta) \) and the estimate (22), we obtain \( \|([AVD(\Delta)]^{-1}\Xi(\Delta)) \leq C_2 \Delta \) for \( \Delta \in [0, a_1] \). This, together with the continuity of the determinant, implies that there exists a number \( a > 0 \) such that \( \det ([AVD(\Delta)]^{-1}\Psi(\Delta)) \neq 0 \) if \( 0 < \Delta \leq a \). Consequently, \( \Psi(\Delta) \neq 0 \), \( 0 < \Delta \leq a \). Taking into account the definition of the matrix \( \Psi(\Delta) [\text{see formulas (19) and (21)}] \) and the properties of determinants, we obtain \( |\det \Psi(\Delta)| = |\det \partial \varphi(\bar{x}_0, 0)/\partial u| \), which completes the proof of the lemma.

**Proof of the theorem.** Suppose that the assumptions of Lemmas 1 and 2 are satisfied. We take a \( \Delta_0 \) such that \( 0 < \Delta_0 \leq \min \{a, \tau_0/n\} \), where \( a \) is the number defined in Lemma 2 and \( \tau_0 \) is the number occurring in the definition of a strongly nonwandering point. We construct the mapping \( \mathcal{F}(x, u) = \mathcal{F}_{\Delta_0}(x, u) \) and apply the implicit function theorem [8, p. 488] to it.

By the theorem on the smooth dependence of the solution of the Cauchy problem on the initial data [6, p. 120], we have \( \bar{x}(-t_n, \bar{x}_0) \in C^{1,1}_x \). Lemma 1 implies that \( x(t_n, u, x_0) \in C^{1,1}_{u, x_0} \). Hence \( \mathcal{F}(x, u) = x(t_n, u, \bar{x}(-t_n, x)) - \bar{x}_0 \in C^{1,1}_{u, x_0} \). By the definition of the mapping \( \mathcal{F} \), we have \( \mathcal{F}(\bar{x}_0, 0) = 0 \). Lemma 2, together with the choice of \( \Delta_0 \), implies that \( \det \partial \mathcal{F}(\bar{x}_0, 0)/\partial u \neq 0 \).
Therefore, the assumptions of the implicit function theorem are satisfied. By this theorem, there exist numbers \( \alpha > 0 \) and \( \beta > 0 \) and a function \( \varphi : B^\alpha (\bar{x}_0, \alpha) \to B^\beta (0, \beta) \), \( \varphi \in C^1 \), such that \( \mathcal{F}(\mathbf{x}, \mathbf{u}) = 0 \) if and only if \( \mathbf{u} = \varphi(\mathbf{x}) \) for any \( (\mathbf{x}, \mathbf{u}) \in B^\alpha (\bar{x}_0, \alpha) \times B^\beta (0, \beta) \). In our case, this means that for each \( \mathbf{x} \in B^n (\bar{x}_0, \alpha) \), the system can be brought from the state \( \bar{x} (-t_n, \mathbf{x}) \) to the state \( \bar{x}_0 \) by the control \( \epsilon \) determined by the vector \( \mathbf{u} = \varphi(\mathbf{x}) \) in time \( n\Delta \).

Since \( \varphi (\bar{x}_0) = 0 \), we have \( \|\mathbf{u}\| = \|\varphi(\mathbf{x}) - \varphi (\bar{x}_0)\| \). The function \( \varphi \) is continuous; therefore, for any \( \varepsilon > 0 \), there exists a \( \delta (\varepsilon) > 0 \), \( \delta (\varepsilon) \leq \alpha \), such that \( \|\mathbf{u}\| < \varepsilon \) for \( \|\mathbf{x} - \bar{x}_0\| < \delta (\varepsilon) \). Here \( \| \cdot \| \) is the Euclidean norm in \( R^n \).

Let us choose some \( \varepsilon > 0 \). Since \( \bar{x}_0 \) is strongly nonwandering, it follows that there exists a \( \tau_\varepsilon > \tau_0 > n\Delta_0 \) such that \( \|\bar{x}(\tau_\varepsilon, \bar{x}_0) - \bar{x}_0\| < \delta (\varepsilon) \). We set

\[
u_\varepsilon(t) = \begin{cases} 0 & \text{for } t \in [0, \tau_\varepsilon - n\Delta_0), \\ u_i & \text{for } t \in [\tau_\varepsilon - (n - i)\Delta_0, \tau_\varepsilon - (n - 1)\Delta_0), \quad i = 0, \ldots, n - 1, \end{cases}
\]

where \( \mathbf{u} = [u_i]_{i=0}^{n-1} = \varphi (\bar{x}(\tau_\varepsilon, \bar{x}_0)) \).

By the definition of \( \varphi \) and the construction of \( u_\varepsilon(t) \), relation (6) is valid. The validity of (5) follows from the choice of \( \tau_\varepsilon \). The proof of the theorem is complete.

**Remark.** The assertion of the theorem remains valid if condition (2) is replaced by the following one:

\[ (2_0) \quad G_\Omega \in C^{n,1}_{\mathbf{x}, \mathbf{u}}, \text{ where } \Omega \text{ is a neighborhood of the point } (\bar{x}_0, 0) \in R^{n+1} \text{ and } F_\Omega \text{ is the restriction of } F \text{ to } \Omega. \]

The proof of the theorem with this new condition remains the same with the only difference that the mapping \( \mathcal{F}(\mathbf{x}, \mathbf{u}) \) is defined not for all values of the arguments but on the set \( \Delta \in (0, \delta_1] \), \( (\mathbf{x}, \mathbf{u}) \in B^{2n} ((\bar{x}_0, 0), \delta_2) \), where \( \delta_1 \) and \( \delta_2 \) are positive numbers such that

\[ (x(t, \mathbf{u}, \bar{x}(-t_n, \mathbf{x})), u(t)) \in \Omega, \quad t \in [0, t_n), \]

for \( u(t) \) of the form (4). The existence of such numbers can be proved with the use of the integral continuity theorem [6, p. 116].

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