Parametric roll resonance and energy transfer

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Traineeship report

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Abstract

Parametric roll resonance on a ship is a condition where a large roll motion develops rapidly in moderate head or following seas. It is a nonlinear oscillation which belongs to a class of systems called *parametrically excited systems*. The phenomenon is investigated by considering three ship models of which the dynamical properties and the transfer of energy between their different modes is assessed.

The models are derived from basic mechanical systems consisting of a mass mounted on a spring with a pendulum mounted to the mass. The equations of motion are derived and the stability properties of their solutions are investigated. From this analysis the conditions for parametric roll resonance to occur are derived.

Based on the results of the analysis of the dynamics numerical simulations are conducted and the transfer of energy between the different modes of the ship models during parametric resonance is evaluated. A saturation phenomenon occurs and the transfer of energy becomes evident. The selected models cannot reproduce the very complex phenomenon of ship-fluid interaction in great detail, but are physically correct in so far as both the restoring and the coupling effects, which constitute the governing terms in the dynamics of a vessel, are considered.
Preface

Parametric roll receives an increasing amount of attention during the last decade. Some accidents happened in 1998 and onwards with large container ships and these triggered the marine community to investigate the phenomenon and come with solutions to detect or predict the resonance conditions and stabilize the roll motion. The Technical University of Denmark is involved in this research, and as part of my Masters programme in Mechanical Engineering at the Technical University of Eindhoven, an abroad traineeship was offered to learn more about the phenomenon. This report was prepared during this abroad traineeship at the Department of Electrical Engineering at the Technical University of Denmark.

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Nomenclature

- $\tau_{4,hs}$ - roll restoring moment
- $\chi$ - encounter angle of the wave
- $\lambda$ - wave length
- $\omega_w$ - encounter frequency of the wave
- $\omega_e$ - wave frequency
- $\rho$ - water density
- $GM_a$ - amplitude of the variation in metacentric height in waves
- $GM$ - metacentric height in calm water
- $\nabla$ - displaced water volume
- $g$ - gravitational acceleration
- $h_w$ - wave height
- $L_{pp}$ - ship length between perpendiculars
- $S$ - intercepted water plane area
- $T_e$ - wave encounter period
- $T_{\phi}$ - roll natural period
- $U$ - ship forward speed
Chapter 1

Introduction

1.1 Parametric roll resonance

Parametric roll resonance on a ship is a condition where large roll motion develops rapidly in moderate head or following seas. It is a nonlinear oscillation which belongs to a class of systems called *parametrically excited systems*. In contrast with forced oscillations, in which the excitations appear as inhomogeneities in the governing differential equations, with parametrically excited systems the excitations appear as time-varying coefficients in the equations of motion. The large roll motion during parametric resonance onsets completely unexpected and the vessel reaches large roll angles in 5 to 15 roll periods. At first the phenomenon was considered a minor problem for small vessels with marginal stability, like fishing vessels for example. Recently though, some incidents have occurred that proved parametric roll resonance can threaten also the largest of vessels.

In October 1998, the APL China, a post-Panamax containership, was overtaken by a violent storm in the North Pacific Ocean [6]. Port and starboard rolls as great as 35° and 40° were reported to have occurred. Of the almost 1300 on-deck containers, one-third, with their cargoes, had been lost overboard. Lawyers estimated the lost cargo was worth more than the value of the ship, more than 50 million dollar.

In January 2003, the Maersk Carolina, a Panamax container vessel, encountered a storm in the North Atlantic Ocean and roll angles up to 47° have been reported [3]. During this incident, numerous containers were damaged or lost overboard. Cargo claims exceeded 4 million dollars. The vessel itself sustained moderate structural damage.

The mentioned incidents were a turning point in the research on parametric roll resonance. Investigators pointed out that particular hull shapes, flat transom stern and significant bow flare, are more susceptible to parametric roll due to the large stability variations these vessels undergo during wave passage in head seas. Diverse directions in research were then taken.
1.2 Literature review

1.2.1 Parametric roll uncovered

The first observations of parametric resonance phenomena date back to 1831. Faraday reported that surface waves in a fluid-filled cylinder under vertical excitation exhibited twice the period of the excitation itself [5]. On ships, parametric resonance was first observed by Froude in 1861 [7]. He reported that a vessel, whose natural frequency in heave and pitch is twice its natural roll frequency, shows undesirable sea keeping characteristics, which can lead to the possibility of exciting large roll oscillations. From that moment on parametrically induced roll in longitudinal seas has been investigated by the marine research community. In 1868, a mathematical description of parametric resonance was given by Mathieu [15].

The phenomenon of parametric roll can be established with different mechanisms. The first ones were described by Grim in 1952, and Kerwin in 1955 [9, 12]. They concluded that the periodic change in the metacentric height of a vessel, as a consequence of the travelling of the wave along the ship, is principal equivalent to the parametrically induced swings of a pendulum when the pivoting point is oscillating up and down. This in turn allowed them to rewrite the roll equation as Mathieu’s equation.

A nonlinear coupling between heave-roll or pitch-roll motions respectively was described in 1973 by Nayfeh et al. [17]. The coupling becomes more and more effective close to resonant conditions for the vertical motions, thus allowing an energy transfer to the transversal motion. Twenty years later, starting from the same nonlinear roll model of a ship, Sanchez and Nayfeh identified the regions of parameter space where the trivial solutions of the roll equation lose their stability [20]. Oh et al. continued this research adding a third degree of freedom, namely pitch [19]. The authors emphasized the possibility that the energy supplied to heave and pitch by the wave motion could be transferred to the roll motion through nonlinear couplings among these three modes.

More recent France et al. presented a detailed analysis of the incident of the post-panamax containership APL China in October 1998 [6]. The authors were able to assess with no doubts that head sea parametric resonance was the cause of the massive losses suffered by APL China. The comparison between theory based on Mathieu’s equation and model tests allowed France et al. to list four major conditions for parametric roll to occur:

1. the roll natural period is approximately equal to twice the period of the encounter wave, \( T_{\phi} \approx 2T_e \),
2. the wave length is approximately equal to the ship length, \( \lambda_w \approx L_{pp} \),
3. the wave height is greater than a critical level, \( h_w > h_s \),
4. the ship’s roll damping is low.

Moreover, the authors pointed out that particular hull shapes, flat transom stern and significant bow flare, determine a higher susceptibility to parametric roll due to the large stability variations these vessels undergo during a wave passage.

The APL China disaster pushed the classification societies and the marine industry to investigate on parametric roll resonance in order to find adequate strategies and technologies capable of dealing with it.
1.2.2 Recent research directions

The development of highly accurate models for studying the roll motion in parametric resonance has been the primary area of investigation of the involved marine community after the accident with the APL China container ship. Firstly models of low and medium complexity have been proposed for investigating the onset conditions of parametric roll and assessing the stability conditions behind the triggering circumstances. There are for example 1-DOF models where the roll motion is rewritten as the Mathieu equation and stability conditions have been derived starting from Floquet theory [21]. Another example are 3-DOF models where heave and pitch are full nonlinearly coupled [8, 23]. Secondly there are very complex numerical 5 or 6-DOF models where the whole ship dynamics is taken into account for assessing the susceptibility of specific hull forms to parametric resonance [6, 21].

The prediction and risk reduction of parametric roll is another aspect coming under the attention of the Marine community. An approach that aimed at assessing the statistical properties of the roll motion in parametric resonance and likelihood of occurrence was proposed by Belenky, Levadou and Palazzi [1, 14]. Belenky found the roll motion in parametric resonance to be not normally distributed. Levadou and Palazzi combined weather statistics with a parametric roll database, and made a first attempt to quantify the operational risk of parametric resonance. The influence of vessel forward speed on the probability of occurrence of parametric roll was studied by Jensen et al. and Ribeiro e Silva et al [11, 4]. They concluded that the probability of experiencing large roll angles decreases significantly when the ship forward speed is increased or decreased during parametric resonance. This agrees with the theoretical expectation since varying the forward speed affects the wave encounter frequency and can detune the resonance condition.

The susceptibility to parametric roll of different vessel types is also an aspect receiving extra attention after the APL China accident. By means of approximations of the first region of instability of the Ince-Strutt diagram associated with Mathieu's equations, Shin et al. derived the susceptibility criteria of parametric resonance to occur in a deterministic approach [21]. Spyrou et al. made an evaluation of these susceptibility criteria and found that these analytical formulae can successfully characterize parametric roll described as a Mathieu-type system [22]. The prediction potential decreases however when the mathematical model becomes more complex and takes into account for example a non harmonic variation of the roll restoring moment.

Finally, the use of anti-rolling devices has currently been researched by the Marine industrie. The use of passive anti-roll tanks to increase damping in the roll equation was proposed by Shin et al. [21]. From a numerical study the authors pointed out that applying passive U-tanks might be an effective technology to lessen the intensity of parametric roll resonance. Also sponsons were proven to be of limited effectiveness of reducing roll oscillation during resonance. Galeazzi investigated the use of fin-stabilizers combined with speed control to stabilise parametric roll resonance, which seems to be very effective [8]. These solutions however introduce difficulties with berthing and loading or occupy a great deal of onboard space and thereby reduce the amount of containers.
1.3 Aims

In the previous sections it was made clear that already a lot of research has been performed on parametric roll. Since Mathieu gave a mathematical description of parametric resonance, numerous models of varying complexity have been developed to study the phenomenon. Also various authors have described that the energy supplied by the wave motion to heave and pitch could be transferred to the roll motion through nonlinear couplings between these three modes. The focus of this assignment will be the investigation of this energy transfer by assessing various mathematical ship models. Therefore, the aim of this research assignment is twofold.

1. First the dynamics of several ship models are analysed.
2. Secondly the transfer of energy between the different modes of these models is analysed.

1.3.1 Analysis of the dynamics

With the analysis of the dynamics the stability properties of several ship models will be investigated. The fundamental idea is to investigate the most general case of ship motion in a moderate head, regular sea with the possibility of a nonlinear coupling between heave-pitch-roll motions. On this basis, three models are selected to be used in this study.

1. Heave-roll model
2. Pitch-roll model
3. Heave-pitch-roll model

These models describe the different modes in a simplified form and are derived from basic mechanical systems consisting of a mass mounted on a spring with a pendulum attached to the mass. Such simple models cannot reproduce the very complex phenomenon of ship-fluid interaction in great detail, but they are physically correct in so far as both the restoring and the coupling effects, which constitute the governing terms in the dynamics of a vessel, are considered. Some of the equations which describe these models have similarities with the Hill-Mathieu equation. Hence, to initiate the analysis of the dynamics, first the dynamical properties of the Hill-Mathieu equation are assessed. The stability domains of the Hill-Mathieu equation and the three ship models will be analysed and the conditions for parametric roll resonance to occur shall be identified.

1.3.2 Energy transfer analysis

The second aim of this study involves the analysis of the transfer of energy between the different modes of the three presented ship models. Based on the results of the analysis of the dynamics, parametric roll can be simulated with numerical experiments and the transfer of energy can be assessed. Two methods are applied to perform this analysis. On the one hand a saturation phenomenon is investigated. Preceding research
projects describe a saturation phenomenon in the forced response of systems with quadratic nonlinearities in the presence of internal resonance. For example, consider a ship whose motion is restricted to pitch and roll only. When the resonance conditions are met, but only the pitch mode is excited, one expects the roll mode to be dormant. Initially this is true, but when the excitation amplitude is increased and reaches a critical value, the response in pitch reaches a maximum amplitude and the energy "spills over" to the roll mode which then begins to oscillate. On the other hand the transfer of energy is analysed by associating the different terms in the equations of potential- and kinetic energy with one of the particular modes of a system. This way, the transfer of energy between different modes can be evaluated, in particular during parametric resonance.

1.4 Overview of the report

In Chapter 2 the analysis of the dynamics of different ship models is considered. First the basic kinematic properties of ships are explained in order to give a better understanding of the terms used in the maritime industry. Also the origin and physics of parametric roll are explained. After this introduction Chapter 2 continues by discussing the stability properties of a single-degree of freedom system that has familiarities with the multi-degree-of-freedom systems mentioned later on. The dynamical properties are assessed by applying Floquet theory. The definition of autoparametric systems is introduced and the conditions for parametric roll to occur with the three presented ship models are identified.

Chapter 3 deals with the analysis of the transfer of energy between the different modes of the ship models. First the saturation phenomenon and the tuning and detuning of parametric roll resonance is discussed. Thereafter these phenomenon will be investigated by applying the three ship models.

Finally, in Chapter 4.2, the conclusions that are drawn from the presented results are given.
Chapter 2

Analysis of the dynamics

2.1 Ship kinematics

A ship is a rigid body whose motions are described by six degrees of freedom, each of them denoted by a coordinate. Two reference frames are employed to define these coordinates [2]:

1) the $S$-frame is an inertial reference frame and it coincides with the ship fixed coordinates in some initial condition,
2) the $S^*$-frame is a body-fixed frame which is fixed to the ship hull and it moves with the vessel. The origin of this frame and its axes coincides with the principal axes of inertia of the ship.

The $S$-frame is used to determine the position of a vessel and the $S^*$-frame is on the other hand used to express linear and angular velocities. Together they define the orientation of a vessel. In Figure 2.1 a summary is given of the coordinate definitions, and in Figure 2.2 the coordinate and frame definitions are displayed.

![Figure 2.1: Description of ship motions in 6 DOF.](image)

2.2 Parametric roll

In order to elaborate on the phenomenon of parametric roll, first some hydrodynamic definitions are explained. Note that for simplicity a vessel sailing in moderate head, regular seas is considered. In a head sea, the roll mode cannot be excited by the sea motion since forces generated by the wave pressure have no lateral components. Hence only the motions in the vertical plane, heave and pitch, can be
excited. These two motions cause periodic variations of the submerged hull geometry which is in direct relation with the intercepted water plane area $S$, resulting in a periodic variation of the metacentric height $GM$ \[18\]. This influences the stability properties of a vessel through the roll restoring moment. The periodic fluctuation of the metacentric height is given by the following formula:

$$GM(t) = GM + GM_a \cos(\omega_e t),$$

(2.1)

where $GM$ equals the metacentric height in calm water, $GM_a$ is the amplitude of the variations of the metacentric height in waves and $\omega_e$ is the wave encounter frequency. The relation between $GM$ and the roll restoring moment $\tau_{4,hs}$ is, in approximation for small angles, given by:

$$\tau_{4,hs}(t) \approx \rho g \nabla GM(t) \varphi(t),$$

(2.2)

where $\rho$ is the water density, $g$ is the acceleration of gravity, $\nabla$ is the displaced water volume and $\varphi$ is the roll angle. The wave encounter frequency is determined with the following relation:

$$\omega_e = \omega_w - \frac{\omega^2}{g} U \cos \chi,$$

(2.3)

where $\omega_w$ is wave frequency, $U$ is the ship forward speed and $\chi$ is the encounter angle of the wave.

The following 2 situations alternate:

1) a wave through is midships. In this case $S$ is relatively larger than in calm water, inducing a larger restoring moment and thereby an increase in stability,

2) a wave crest is midships. In this case $S$ is relatively smaller than in calm water, causing a smaller restoring moment and thereby a reduction in stability.

Parametric roll occurs due to this periodic variation in the roll restoring moment, which is illustrated in the following example. When a vessel is subject to a periodic motion in roll in head seas, caused by a windgust for example, and the vessel is between a wave crest and trough at amidships position, its response will be greater than in calm water since it is approaching a situation of increased stability. Therefore the ship will roll back to a larger angle than it would have done in calm water. This situation is indicated with $\Delta \phi_1$ in Figure 2.3. Due to inertia it continues to roll toward the other side after it has rolled back to zero degrees. Now the ship encounters a wavecrest amidships however, which determines a reduced restoring moment with respect to calm water. The result is, again, that the ship rolls to larger
Figure 2.3: Development of parametric roll.

angle than it would have done in calm water condition. Therefore the roll angle is increased again over the second quarter of the roll period with respect to the the calm water condition, illustrated by $\Delta \phi_2$ in Figure 2.3. This alternating sequence of instantaneous increased and reduced restoring moment causes the roll angle to keep increasing.

Regarding the equations of motion of a ship, the metacentric height is the time-varying parameter exciting the roll motion [8]. Assuming the metacentric height varies periodically with period $T_e = \frac{2\pi}{w_e}$, the resonance condition becomes $T_\phi = 2T_e$, or in other terms $w_e = 2w_\phi$; the frequency of the fluctuations of $GM$ must be twice the roll natural frequency. In the next sections this is explained to be the principal parametric resonance condition. Note that this relation can be relaxed since the onset of the resonant motion can happen if $w_e \approx 2w_\phi$.

2.3 Single-degree-of-freedom system

The dynamic models which are assessed in the next sections are parametrically excited systems. In parametrically excited systems, the excitations appear as time-varying coefficients in the governing differential equations. In contrast with the case of external excitations, in which a small excitation cannot produce a large response unless the frequency of the excitation is close to one of the natural frequencies of the system, a small parametric excitation can produce a large response when the frequency of the excitation is close to twice one of the natural frequencies of the system. This is called principal parametric resonance.

First the dynamics of a single-degree-of-freedom system are analysed, which will prove to be useful when assessing the behaviour of multi-degree-of freedom nonlinear ship models later on. In order to characterize the behaviour of the linear
single-degree-of-freedom system, the Floquet theory is described briefly [16, 10].

2.3.1 Floquet theory

The general form for a linear homogenous system with periodic coefficients is:

\[ \dot{x} = A(t)x, \quad (2.4) \]

where \( A(t+T) = A(t) \) for all \( t \). For such systems, the following theorem holds:

**Theorem 2.1.** Let \( X(t) \) be a fundamental matrix for \((2.4)\). Then \( X(t+T) \) is also a fundamental matrix, and there exists a non-singular constant matrix \( B \) such that:

\[ X(t+T) = X(t)B \quad \forall t. \quad (2.5) \]

Also:

\[ \det B = \exp \left\{ \int_0^T \text{tr} A(s)ds \right\}. \quad (2.6) \]

Theorem 2.1 shows that in general the fundamental matrix \( X(t) \) for \((2.4)\) is not periodic and, hence, that the general solution of \((2.4)\) is not periodic, in general. The following definition introduces the characteristic exponents, also known as Floquet exponents.

**Definition 2.1.** Let the eigenvalues of \( B \), defined by \((2.5)\), be \( \rho_1, ..., \rho_n \), called the characteristic multipliers for \((2.4)\). The characteristic exponents \( \mu_1, ..., \mu \) are defined by:

\[ \rho_1 = e^{\mu_1 T}, \quad ..., \quad \rho_n = e^{\mu_n T}. \quad (2.7) \]

The characteristic multipliers can take positive and negative values and can be either real or complex. The characteristic multipliers and, hence, the characteristic exponents, do not depend on the particular choice of fundamental matrix \( X(t) \) and are intrinsic properties of \((2.4)\). The consequence of this statement results in the following theorem:

**Theorem 2.2.** Let \( \rho \) be a characteristic multiplier for \((2.4)\) and let \( \mu \) be the corresponding characteristic exponent so that \( \rho = e^{\mu T} \). Then there exists a solution \( x(t) \) of \((2.4)\) such that:

\[ x(t+T) = \rho x(t) \quad \forall t. \quad (2.8) \]

Further, there exists a periodic function \( p(t) \), i.e. \( p(t+T) = p(t) \), such that:

\[ x(t) = e^{\mu t}p(t). \quad (2.9) \]

Now the structure of the general solution of \((2.4)\) can be determined. It follows from Theorem 2.2 that there exist \( n \) linearly independent solutions of \((2.4)\), given by:

\[ x^n(t) = e^{\mu_n t}p^n(t) \quad (n = 1, ..., n), \quad (2.10) \]
where \( p(t) \) is a periodic function with period \( T \). The key components here are the factors \( e^{\mu t} \), since, it are these factors which determine the long term behaviour of solutions. Therefore by identifying the sign of the characteristic exponent \( \mu \) it is possible to infer the stability of the solution. It can be computed by inverting (2.7):

\[
\mu_n = \frac{1}{T} \ln (\rho_n) \quad (n = 1, \ldots, n).
\]

In the next section, this theory will be applied to assess the stability of the solutions of a single-degree-of-freedom system.

### 2.3.2 The Hill-Mathieu equation

Consider the second order differential equation:

\[
\ddot{x} + a(t) x = 0,
\]

(2.12)
discussed first by Hill (1886) [10]. Hence, it is called Hill’s equation. When:

\[
a(t) = \delta + 2\epsilon \cos 2t,
\]

(2.13)
equation (2.12) reduces to:

\[
\ddot{x} + (\delta + 2\epsilon \cos 2t) x = 0,
\]

(2.14)
which was discussed by Mathieu (1868) and (2.14) is referred to as Mathieu’s equation. This equation governs the response of many physical systems to a sinusoidal parametric excitation. An example is a pendulum whose point of support is made to oscillate along a vertical line, as shown in Figure 2.4 [10, 23]. In this example, \( x \) from (2.14) complies with the angle \( \theta \) between the pendulum and the vertical, \( \delta = \sqrt{g/l} \) and \( 2\epsilon \cos 2t = \zeta'' \).

![Figure 2.4: A pendulum whose support is moving along a vertical line.](image)

Floquet theory is now applied to assess the stability of the solutions of (2.14) by computing the eigenvalues of constant matrix \( B \). First (2.14) is rewritten as an equivalent first order system:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-a(t) & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix},
\]

(2.15)
where \( x_1 = x \) and \( x_2 = \dot{x}_2 \). Note that here \( \text{tr} A(t) = 0 \). Next the fundamental matrix \( X(t) \) for (2.15) is formed such that \( X(0) = E \):

\[
X = \begin{bmatrix}
x_1 & \dot{x}_1 \\
x_2 & \dot{x}_2
\end{bmatrix},
\]

(2.16)
where $x_1$ and $x_2$ are linearly independent solutions of equation (2.14) so that $X(0) = I$. The matrix $B$ is given by:

$$B = X^{-1}(0) \, X(T),$$

which results in:

$$B = \begin{bmatrix}
\dot{x}^1(T) & \dot{x}^2(T) \\
x^2(T) & \dot{x}^1(T)
\end{bmatrix}. \quad (2.18)$$

As a consequence of $\text{tr} \, A(t) = 0$, two implications follow:

1) $\det B = 1$.
2) since $X(0) = I$ and the Wronskian of the fundamental matrix (2.16) is constant not equal to zero, which implies linearly independent solutions independent of time,

$$W(t) \equiv \det X = x_1 \dot{x}_2 - x_2 \dot{x}_1 = W(0) = 1. \quad (2.19)$$

The characteristic multipliers, $\rho$, are the eigenvalues of $B$, hence, are given by:

$$\rho^2 - 2\phi \rho + 1 = 0, \quad \text{where} \quad \phi = \frac{1}{2} \left\{ x^1(T) + \dot{x}^2(T) \right\}. \quad (2.20)$$

Thus the eigenvalues $\rho_{1,2}$ are functions of the single parameter $\phi$ and are given by:

$$\rho_{1,2} = \phi \pm \sqrt{\phi^2 - 1}. \quad (2.21)$$

When the real part of one of the eigenvalues is positive, $x$ is unbounded (unstable) with time, while when the real parts of all the eigenvalues are zero or negative, $x$ is bounded (asymptotically stable) with time [13]. Where the real parts of the eigenvalues equal zero, stable motions are separated from unstable motions. Regarding (2.14), $\phi$ is a function of the parameters $\delta$ and $\epsilon$. The locus of transition values of $\delta$ and $\epsilon$ separates the $\delta \epsilon$-plane into regions of stability and instability as shown in Figure 2.5. This graph is known as the Ince-Strutt diagram [16].

![Figure 2.5: Ince-Strutt diagram for the linear Mathieu equation](image)

When $\epsilon = 0$, positive values of $\delta$ correspond to stable positions of the pendulum (i.e., downward position), while negative values of $\delta$ correspond to unstable
positions of the pendulum (i.e., upward position). In the presence of the parametric excitation, Figure 2.5 shows that there are values of $\delta$ and $\epsilon$ for which the downward position is unstable and the upward position is stable.

In the case of $\epsilon = 0$, (2.14) reduces to the linear harmonic oscillator whose general solution is:

$$x(t) = C_1 \cos \sqrt{\delta} t + C_2 \sin \sqrt{\delta} t,$$

which is periodic with period $T_0 = 2\pi/\sqrt{\delta}$. Also, from $X(0) = I$ it follows that:

$$x_1(t) = \cos \sqrt{\delta},$$

$$x_2(t) = \frac{1}{\sqrt{\delta}} \sin \sqrt{\delta}.$$  

Hence, from (2.20) it now follows that:

$$\phi = \cos \sqrt{\delta} T \quad (for \quad \epsilon = 0)$$

The stability boundaries for $\epsilon = 0$ are now given by:

$$\varphi = 1, \quad \sqrt{\delta} T = 2k\pi \quad or \quad T = kT_0,$$

$$\varphi = -1, \quad \sqrt{\delta} T = (2k + 1)\pi \quad or \quad 2T = (2k + 1)\ T_0,$$

where $k = 0, 1, 2, \ldots$. These are conditions for parametric resonance. When $\varphi = -1$ and $k = 0$ the Mathieu equation is in the resonance condition called principal parametric resonance, which is characterized by $T_0 = 2T$. In other words, the forcing frequency is twice the natural frequency. By this means, large amplitude oscillations can be generated with only a very small amplitude of forcing. In Figure 2.6 two responses of the Mathieu equation in different resonance conditions are displayed for equal values of $\epsilon$, but at different values of $\delta$, that is $\delta = 1$ and $\delta = 4$. The amplitude of the response in principal parametric resonance grows dramatically in contrast to the response in the second unstable region.

In almost all physical systems dissipation is present. This has a stabilizing effect on all single-degree-of-freedom systems. Including a viscous term, (2.14) is rewritten as:

$$\ddot{x} + 2\mu x + (\delta + \epsilon \cos 2t) u = 0.$$  

Figure 2.6: Response of the linear Mathieu equation.
The transition curves separating stable from unstable solutions of (2.26) are shown in Figure 2.7. Comparing Figure 2.7 with Figure 2.5 shows that the addition of viscous damping lifts the unstable regions from the δ-axis, rounds the point at the bottom, and narrows the unstable regions.

![Figure 2.7: Ince-Strutt diagram of the linear damped Mathieu equation.](image)

### 2.4 Autoparametric systems

The multi-degree-of-freedom systems considered in the next sections of this chapter are classed as autoparametric systems [23]. According to this definition autoparametric systems are vibrating systems of at least two constituting subsystems; a primary- and a secondary system. The primary system is generally in a vibrating state and can be externally forced, self-excited, parametrically excited or a combination of these. The secondary system is nonlinearly coupled to the primary system such that it can be at rest while the primary system is vibrating. This state is called the semitrivial solution or normal mode. With this definition, autoparametric systems can be characterized as follows:

1. an autoparametric system consists of at least a primary system, with states $x$ and $\dot{x}$ for example, coupled with a secondary system, with states $y$ and $\dot{y}$ for example,
2. there exists a semitrivial solution, with the property:

   $$\sum_{i=1}^{N} \left[ x_i^2(t) + \dot{x}_i^2(t) \right] \neq 0 \quad \forall t,$$

   $$y_i(t) = \dot{y}_i(t) = 0 \quad \forall t, \quad i = 1, ..., n,$$

   \hspace{1cm} (2.27)

3. in certain intervals of the frequency of the excitation the semitrivial solution can become unstable,
4. in or near the instability intervals of the semitrivial solution there is autoparametric resonance; the vibrations of the primary system act as a parametric excitation of the secondary system, which will no longer remain at rest.

This definition shows that in studying autoparametric systems, the determination of stability and instability conditions of the semitrivial solutions is always the
initial focus. Since the three systems considered in the next sections are autoparametric systems, the analysis of these systems is therefore started with finding the semitrivial solutions and assessing their stability. Since the secondary systems are always parametrically excited, the stability of semitrivial solutions can be assessed using the same techniques as used in the analysis of the Hill-Mathieu equation in the previous section.

In Chapter 3 the energy transfer between the different modes of the autoparametric systems will be investigated, which is a transfer between the primary- and secondary system.

### 2.5 Heave-roll model

In this section the model displayed in Figure 2.8 is considered. It consists of a pendulum attached to a mass which is restrained by an elastic spring and damper. Herein is $z$ the relative vertical displacement of the mass $m_1$, $\varphi$ is the angular displacement of the pendulum-mass $m_2$, $k$ is the elastic constant of the spring, $l$ is the length of the rod, $b$ and $c$ are the damping coefficients of the linear and angular motions, respectively. The system is forced to oscillate sinusoidally in the vertical direction by means of a periodic force with amplitude $\alpha$ and frequency $\omega$.

This system is able to simulate the dynamical behaviour of a vessel running in a moderate head, regular sea and gives the possibility of reproducing the nonlinear coupling between heaving and rolling motions. Herein the vertical motion of the mass corresponds to heave and the motion of the pendulum corresponds to roll. The coupling among the oscillations is accomplished by connecting the two masses, and the effect of the waves is simulated by means of external forcing. A moderate sea means there does not come any water on the deck and no slamming takes place.

The equations of motion for this system, reduced to dimensionless form, are:

\[
\ddot{w} + \kappa \dot{w} + q^2 w + \mu \left( \dot{\varphi} \sin(\varphi) + \dot{\varphi}^2 \cos(\varphi) \right) = a\eta^2 \cos(\eta \tau),
\]

\[
\ddot{\varphi} + \kappa_0 \dot{\varphi} + \sin(\varphi) + \left( \ddot{w} - a\eta^2 \cos(\eta \tau) \right) \sin(\varphi) = 0. \tag{2.28}
\]
The derivation and reduction is shown in Appendix A. Note that the following new time variable is introduced:

\[ \tau = (g/l)^{\frac{1}{2}} t. \]  

(2.29)

This implies that

\[ \dot{w} = \frac{dw}{d\tau} = (g/l)^{\frac{1}{2}} \frac{dw}{dt}. \]  

(2.30)

The semitrivial solution of (2.28) has the following form:

\[ w_0(\tau) = a \left( A \cos(\eta \tau) + B \sin(\eta \tau) \right), \]

\[ \varphi_0(\tau) = 0. \]  

(2.31)

By substituting (2.31) into (2.28) A and B are determined, which are:

\[ A = \frac{\eta^2 (q^2 - \eta^2)}{(q^2 - \eta^2)^2 + (\kappa \eta)^2}, \]

\[ B = \frac{\kappa \eta^2}{(q^2 - \eta^2)^2 + (\kappa \eta)^2}. \]  

(2.32)

The stability of the steady state solutions (2.31) can be obtained by substituting small perturbations into (2.28), by letting:

\[ w = w_0 + u, \]

\[ \varphi = \varphi_0 + \psi. \]  

(2.33)

This leads to the resulting perturbation equations, which are, at a first-order approximation:

\[ \ddot{u} + \kappa \dot{u} + q^2 u = 0, \]

\[ \ddot{\psi} + \kappa_0 \dot{\psi} + \psi - a \eta^2 [(1 + A) \cos(\eta \tau) + B \sin(\eta \tau)] \psi = 0. \]  

(2.34)

The first equation of (2.34) is a homogenous differential equation with constant coefficients and its solution is asymptotically stable. The second equation is a Mathieu equation and its stability depends on the values taken for the different parameters. The first instability region, where the solution may be approximated by (2.22), can be obtained by inserting (2.22) into the second equation of (2.34). The nontrivial solution for \( C_1 \) and \( C_2 \) results in:

\[ \left( 1 - \frac{1}{4} \eta^2 \right)^2 + \frac{1}{4} \kappa_0^2 \eta^2 - \frac{1}{4} a^2 \eta^4 \left[ (1 + A)^2 + B^2 \right] = 0 \]  

(2.35)

After a reordering of terms of (2.35), the instability threshold for the appearance of parametric resonance may be expressed in the following explicit form:

\[ a_c = \frac{2}{\eta^2} \sqrt{\left( 1 - \frac{1}{4} \eta^2 \right)^2 + \frac{1}{4} \kappa_0^2 \eta^2 \left( (1 + A)^2 + B^2 \right)} \]  

(2.36)

where \( a_c \) is the critical amplitude. When the excitation amplitude \( a \) is larger than \( a_c \) parametric resonance will occur. In Figure 2.9 the instability threshold \( a_c \) is
depicted as a function of $\eta$ and $q$. To obtain a more convenient representation, the direction of the $a_c$-axis has been changed so that minima appear as maxima and the instability domain now lies below the surface. The following values have been used in this case: $\kappa = 0.1$, $\kappa_0 = 0.1$. The diagram shows that the possibility for the appearance of parametric resonance is increased when the frequency ratio between the excitation and the roll-motion equals two and, also, when $\eta = q$, i.e. the heave motion becomes resonant. These conditions will be used in Section 3.1 for the analysis of the energy transfer during parametric roll resonance. In the next section a pitch-roll model is considered.

Figure 2.9: Instability threshold $a_c$ for heave-roll model, instable is under surface.

### 2.6 Pitch-roll model

The forced response of a ship whose motion is restricted to pitch and roll only can be modeled by the following equations \cite{16, 17}:

\begin{align}
\ddot{\theta} + \mu_1 \dot{\theta} + \theta + \alpha_1 \varphi^2 &= F \eta^2 \cos (\eta \tau), \\
\ddot{\varphi} + \mu_2 \dot{\varphi} + q^2 \varphi + \alpha_2 \theta \varphi &= 0. \tag{2.37}
\end{align}

Here $\theta$ is the pitch angle, $\varphi$ is the roll angle, $q = \omega_1/\omega_2$ is the ratio between the natural frequencies of the two systems, $\mu_n$ are the linear damping coefficients, $\alpha_n$ are the nonlinear coupling coefficients, $F$ is the amplitude of the external forcing, and $n = 1, 2$. Note that the nonlinear coupling of the pitch motion through $\varphi^2$ in (2.37) is claimed by Nayfeh et al. \cite{16, 17}, but other researchers have been unable to confirm this coupling. Motivated by previous research Nayfeh et al. adopted the $\varphi^2$-term in the pitch-roll model and simulation results seemed to agree with real measurements. In contrast with this, a coordinate transformation described by Blanke et al. \cite{2} for example points out there seems to be a coupling through $\dot{\varphi}^2$ instead of $\varphi^2$.

It is also claimed that the described system is derived from the mechanism displayed in Figure 2.10. This is also unconfirmed, though in this report the model is used to obtain the equations of kinetic- and potential energy in order to assess
the energy transfer of (2.37) in Section 3.2. The mechanism consists of a rigid block with mass \( m_1 \) which is connected to a rigid vertical wall via a linear spring, with spring constant \( k \), and a viscous damper with damper constant \( \mu_1 \). The rigid block is subjected to a periodic force \( F\eta_2 \cos (\eta t) \). A pendulum is connected to the rigid block at height \( h \) via a hinge with viscous damping with damping constant \( \mu_2 \). The angle of rotation of the pendulum with respect to the \( x_2 \)-axis is defined as \( \varphi \). The pendulum consists of a massless rod with at the free end a concentrated mass \( m_2 \) attached to it. The derivation of the equations for the kinetic- and potential energy is given in Appendix C. The equations of motion for this mechanism are:

\[
\begin{align*}
(m_1 + m_2) \ddot{\theta} + m_2 l \dddot{\varphi} \cos (\varphi) - m_2 l \ddot{\varphi}^2 \sin (\varphi) + k \theta + \mu_1 \dot{\theta} &= F\eta_2 \cos (\eta t), \\
+m_2 l \ddot{\theta} \cos (\varphi) + m_2 l^2 \dddot{\varphi} + m_2 g \sin (\varphi) + \mu_2 \dot{\varphi} &= 0. \quad (2.38)
\end{align*}
\]

Figure 2.10: Mass-pendulum system for derivation of pitch-roll model, according to Nayfeh et al. [16, 17].

To continue with the dynamical analysis the semitrivial solution of (2.37) is assessed, which is given by:

\[
\begin{align*}
\theta (\tau) &= R \cos (\eta \tau + \psi), \\
\varphi (\tau) &= 0 \quad (2.39)
\end{align*}
\]

where \( R \) is a dimensionless parameter which is obtained by substituting (2.39) into (2.37), and is given by:

\[
R = R_0 = \frac{F\eta^2}{\sqrt{(1 - \eta^2)^2 + \mu_1^2 \eta^2}}. \quad (2.40)
\]

Note that when \( \mu_1 = O (\epsilon) \) and \( \eta = 1 + O (\epsilon) \), the amplitude of the solution is \( R_0 = O (F/\epsilon) \). This situation is related to the main resonance of the primary system. The stability of the steady state solutions (2.39) can be obtained by substituting perturbation solutions into (2.37), i.e. by inserting the expressions:

\[
\begin{align*}
\theta &= R_0 \cos (\eta \tau + \psi) + u, \\
\varphi &= 0. \quad (2.41)
\end{align*}
\]

into (2.37). This yields, in linear approximation:

\[
\begin{align*}
\ddot{u} + \mu_1 \dot{u} + u &= 0, \\
\ddot{v} + \mu_2 \dot{v} + [q^2 + \alpha_2 R_0 \cos (\eta \tau + \psi)] v &= 0. \quad (2.42)
\end{align*}
\]
The solution \( u = 0 \) of the first equation of (2.42) is asymptotically stable. This means the second equation of (2.42) fully determines the stability of the semitrivial solution. This is a Mathieu type equation, and its main instability domain is found for values of \( q \) near \( \frac{1}{2} \eta \). The boundary of this instability domain is given by:

\[
\sigma^2 + \frac{1}{4} \mu^2 \eta^2 - \frac{1}{4} \alpha^2 R_0 = 0,
\]

(2.43)

where

\[
\sigma = q^2 - \frac{1}{4} \eta^2.
\]

(2.44)

When (2.40) is inserted into (2.43), the critical value of excitation amplitude \( F \) can be obtained. This is the value of \( F \) for which the solutions become unstable and defines an instability threshold for the appearance of parametric resonance. It can be expressed in the following form:

\[
F_c = 2 \sqrt{(1 - \eta^2)^2 + \mu^2 \eta^2} \sqrt{\sigma^2 + \kappa^2 \eta^2}.
\]

(2.45)

For values of \( F \) above this critical value \( F_c \), the solution (2.39) is unstable and a non-trivial solution appears. In Figure 2.11 the instability threshold \( F_c \) is displayed as a function of the forcing frequency \( \eta \) and the ratio of natural frequencies \( q \). To obtain a more convenient representation, the direction of the \( F \)-axis has been changed so that minima appear as maxima and the instability domain now lies below the surface.

The graph shows that close to \( \eta = 1 \) and \( q = \frac{1}{2} \eta \) the instability threshold exhibits local minima and the possibility for parametric roll resonance to occur is increased. In Section 3.2 these conditions are employed to analyse the transfer of energy between the pitch and roll modes during parametric roll. In the next section a slightly more complex model will be assessed with three degrees-of-freedom.

![Figure 2.11: Instability threshold \( F_c \) for pitch-roll model, instable is under surface.](image-url)
2.7 Heave-pitch-roll model

This section considers the system shown in Figure 2.12. It consists of two masses restrained by elastic springs supporting two equal pendulums rigidly connected by means of a weightless rod. The masses are excited at the same frequency and amplitude, but experience a certain phase lag to consider the delayed effects of the wave propagating along the ship. This system basically doubles the mechanical model considered in Section 2.5. The different parameters are defined as follows:

![Figure 2.12: Model for simulating heave-pitch-roll motion of a ship in longitudinal waves.](image)

- $z_1$ and $z_2$ are the relative vertical displacements of the masses $m_1$ and $m_2$ respectively,
- $\varphi$ is the angular displacement of pendulum masses $m_3$ and $m_4$,
- $\psi$ is the phase lag,
- $k_1$ and $k_2$ are the elastic constants of the springs,
- $l$ is the length of the rods, $b_1$, $c$ are the damping coefficients of the linear and angular motions, respectively, and $\alpha$ and $\omega$ are the amplitude and the frequency of the excitation.

This mechanical system was developed to simulate the most general case of ship motion in a longitudinal or oblique sea with the possibility of nonlinear coupling between heave-pitch roll motions [23]. The motion of the two rigidly connected pendulums corresponds to roll, and the two masses in vertical motion will reproduce the simultaneous heave-pitch oscillations.

The equations of motion for this system, reduced to dimensionless form, are:

\[
\ddot{w}_1 + \kappa_1 \dot{w}_1 + q_1^2 w_1 + \mu_1 (\varphi \sin \varphi + \dot{\varphi}^2 \cos \varphi) = a \eta^2 \cos (\eta \tau), \\
\ddot{w}_2 + \kappa_2 \dot{w}_2 + q_2^2 w_2 + \mu_2 (\varphi \sin \varphi + \dot{\varphi}^2 \cos \varphi) = a \eta^2 \cos (\eta \tau - \psi), \\
\ddot{\varphi} + \kappa_0 \dot{\varphi} + \sin \varphi + \frac{1}{2} [\ddot{w}_1 - a \eta^2 \cos \eta \tau + \ddot{w}_2 - a \eta^2 \cos (\eta \tau - \psi)] \sin \varphi = 0. \tag{2.46}
\]

The derivation and reduction is shown in Appendix E. The steady-state solution of (2.46) is:

\[
w_{0j}(\tau) = a (A_j \cos \eta \tau + B_j \sin \eta \tau), \\
\varphi_0(\tau) = 0. \tag{2.47}
\]

where $A_j$ and $B_j$, are dimensionless parameters which are obtained by substituting
(2.47) into (2.46):

\[
A_1 = \frac{\eta^2(q_1^2 - \eta^2)}{(q_1^2 - \eta^2)^2 + (\kappa_1\eta)^2}, \\
A_2 = \frac{\eta^2 \left((q_2^2 - \eta^2) \cos(\psi) - \kappa_2\eta \sin(\psi)\right)}{(q_2^2 - \eta^2)^2 + (\kappa_2\eta)^2}, \\
B_1 = \frac{\kappa_1\eta^3}{(q_1^2 - \eta^2)^2 + (\kappa_1\eta)^2}, \\
B_2 = \frac{\eta^2 \left((q_2^2 - \eta^2) \sin(\psi) + \kappa_2\eta \cos(\psi)\right)}{(q_2^2 - \eta^2)^2 + (\kappa_2\eta)^2}.
\]

(2.48)

The stability of the steady state solutions (2.47) can be obtained by substituting small perturbations into (2.46), in the forms:

\[
w_j = w_{0j} + u_j, \\
\varphi = \varphi_0 + \psi.
\]

(2.49)

The resulting first-order approximation equations are:

\[
\ddot{u}_j + k_j \dot{u}_j + q_j^2 u_j = 0, \\
\psi + \kappa_0 \dot{\psi} + \psi - \frac{1}{2} \eta^2 \left(E \cos(\eta \tau) + F \sin(\eta \tau)\right) \psi = 0,
\]

(2.50)

where \(j = 1, 2, E = 1 + A_1 + A_2 + \cos(\psi), \) and \(F = B_1 + B_2 + \sin(\psi)\). The first two equations of (2.50) are homogenous differential equations with constant coefficients and their solutions are asymptotically stable. The third equation is a Mathieu equation and its stability depends on the values of the different parameters. The first instability region, where the solution may be approximated by (2.22), can be obtained by inserting (2.22) into the second equation of (2.50). After reordering of terms of the nontrivial solution for \(C_1\) and \(C_2\) results in the instability threshold for the appearance of parametric resonance:

\[
a_c = \frac{1}{\eta^2} \left[ \left(1 - \frac{1}{4} \eta^2\right)^2 + \frac{1}{2} \kappa_0^2 \eta^2 \right].
\]

(2.51)

In Figure 2.13 the instability threshold \(a_c\) is depicted as a function of \(\eta\) and \(\psi\). To obtain a more convenient representation, again the direction of the \(a_c\)-axis has been changed so that minima appear as maxima and the instability domain now lies below the surface. The following values have been used: \(q_1 = 4, q_2 = 3, \kappa_0 = 0.1, \kappa_1 = 0.1, \kappa_2 = 0.1\).

In general, there are three minima, and these occur for \(\eta\) close to the values 2, 3 and 4. The influence of the phase \(\psi\) on the threshold level is limited to regions of the minima around \(q_1\) and \(q_2\) but becomes substantial in the neighborhood of \(\eta = 2\), where an increase in \(a_c\) occurs for \(\psi \approx \pi\), in other words, when the two masses are excited in anti-phase.

The possibility for the appearance of parametric resonance is increased when the frequency ratio between the excitation and the roll-motion equals two and, also when \(\eta \approx q_1\) or \(\eta \approx q_2\), i.e. the heave motion of one of the two masses becomes resonant. In Section 3.3 the obtained resonance conditions are used to study the transfer of energy between the different modes.
Figure 2.13: Instability threshold $a_c$ for heave-pitch-roll model, instable is under the surface.
Chapter 3

Energy transfer analysis

With the preceding dynamical analysis the stability of the solutions for the three ship models has been investigated and the conditions for parametric roll to occur have been identified. This information will be used in this chapter to assess the transfer of energy between the different modes of the models during parametric resonance.

In Section 2.4 it was explained that the considered ship models are classed as autoparametric systems. The energy transfer will occur between the so-called primary- and secondary systems. When the resonance conditions are met it is expected that a saturation phenomenon will occur. When only the primary system is excited it is expected that the secondary system is dormant. After increasing the excitation amplitude of the primary system to a critical value, the response of the primary system is expected to get saturated and the excess energy "spills over" to the secondary system which then starts to oscillate. This phenomenon will be reproduced by performing various numerical simulations, and the results will be analysed. The method employed here is to run different simulations with a specific ship model in which all parameters are kept equal with the exception of the excitation amplitude which is increased with a small step with each new simulation. The amplitudes of the responses of the different modes will be logged and the saturation phenomenon will become visible.

Another aspect of parametric roll resonance with ships is the phenomenon of detuning the resonance condition by altering the heading or forward speed of the vessel, and hereby changing the wave encounter frequency. This aspect is also investigated in the next sections where numerical simulations are conducted in which the frequency conditions are initially tuned towards the conditions for parametric roll resonance to occur, after which they are detuned again to simulate the change of speed or heading of a ship. The energy transfer during these simulations is assessed by associating the different terms in the equations for kinetic- and potential energy with the primary- or second system.

3.1 Heave-roll model

The model considered here is the heave-roll model from Section 2.5. First the saturation phenomenon will be assessed. In this system this is expected to occur when $\eta$ is near $\omega_2$, but when $a < a_C$, the excitation amplitude of the forcing is too small
for the excitation of the primary system to act as a parametric excitation to the secondary system, i.e. the roll-mode, which is initially nonresponsive and will remain dormant. The stable response of the system will be the semi-trivial solution (2.31). The amplitude of this solution grows linearly with \( a \), as is expected from (2.31) and (2.32). When \( a = a_c \), the semi-trivial solution loses stability, a nontrivial solution appears, and the amplitude of the heave-mode is limited when \( a \) is increased further. The amplitude of the roll-mode however grows over time with increasing \( a \). In other words, the heave-mode becomes saturated and the excess energy "spills over" into the roll-mode. Figure 3.1 illustrates this behaviour. In this graph the black line represents the amplitude of the primary systems’ response, the red line represents the amplitude of the secondary system and the blue line represents the forcing amplitude which is varied from zero to above critical levels. On the basis of 100 numerical simulations the amplitudes of the oscillations of the primary- and secondary system were logged. With every new simulation the excitation amplitude of the primary system was increased with a small increment. The parameters used in these simulations are displayed in the table next to the graph. When \( a \approx 0.23 \), the slope of the amplitude of the primary system starts to decrease and the amplitude of the secondary system starts to increase. This is in accordance with a ship which is advancing in a head or following sea, and, if the waves are big enough and at the right frequency, begins to roll violently.

\[
E_{\text{heave}} = \frac{1}{2} (m_1 + m_2) x_2^2, \quad (3.1)
\]
\[
E_{\text{roll}} = m_2 \left( \frac{1}{2} l^2 x_4^2 + gl \left( 1 - \cos (x_3) \right) \right). \quad (3.2)
\]

These equations allow the evaluation of the amount of energy in the primary-
and secondary-system during each time step of the numerical simulation. These equations are put together by associating the different terms in the equations for potential- and kinetic energy with the primary- or secondary system. Terms which are a function of \( \dot{z} \), or \( x_3 \), are associated with the primary system, and terms which are a function of \( \varphi \) or \( \dot{\varphi} \), or \( x_3 \) and \( x_4 \) respectively, are associated with the secondary system. Since this is a model of a coupled system employing this method is questionable, but for the purpose of evaluating the transfer of energy the results could prove to be quite interesting. Two terms are omitted here; the potential energy from the spring is considered to be associated with the forces acting on the hull generated by the wave pressure, and the nonlinear coupling term being a function of \( \dot{z} \), \( \varphi \) and \( \dot{\varphi} \), or \( x_3 \), \( x_3 \) and \( x_4 \) respectively, is not associated with one particular system. Together, these two omitted terms could account for approximately 10% to 30% of the system’s total energy.

With the state equation a numerical simulation can be conducted in Matlab, and with the given energy functions the energy transfer between the primary- and secondary system can be evaluated. In Figure 3.2, the results are shown for a numerical simulation with the values for the different parameters shown in the table above the graphs. In Appendix B the results are displayed in more detail. In both graphs the horizontal axis represents time, in the left graph the vertical axis corresponds to heave and roll and in the right graph the vertical axis represents energy. The yellow line in the left graph represents the oscillation in heave and the red line represents the oscillation in roll. The black line in the right graph represents the oscillation of energy in the primary system and the yellow line represents the energy in the secondary system. During the simulation \( \eta \) is slowly increased from 1.4 to 2.0 until parametric resonance occurs and a large roll motion develops due to the energy transfer from the primary system to the secondary system. After some time \( \eta \) is decreased to detune the parametric resonance and the roll motion decays to smaller amplitudes. The detuning of the resonance by decreasing \( \eta \), i.e. the excitation frequency or the wave encounter frequency in the case of a vessel, agrees with a change in speed or heading of a vessel.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( \kappa )</th>
<th>( \kappa_0 )</th>
<th>( \mu )</th>
<th>( \alpha )</th>
<th>( n )</th>
<th>( q )</th>
<th>( m_1 ) (kg)</th>
<th>( m_2 ) (kg)</th>
<th>( \varphi ) (m/s²)</th>
<th>( l ) (m)</th>
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<tbody>
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<td>1.0</td>
<td>1.0</td>
<td>0.5</td>
<td>2.0</td>
<td>2.0</td>
<td>1.0</td>
<td>1.0</td>
<td>9.8</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Figure 3.2: Results of numerical simulation of heave-roll model.
3.2 Pitch-roll model

The transfer of energy between the modes of the pitch-roll model from Section 2.6 is considered here. Also in this system a saturation phenomenon can be identified. When $\eta$ is near $\omega_2$ and $F < F_c$, the roll-mode is initially nonresponsive. The stable response of the system will be the semi-trivial solution (2.39). The amplitude of this solution grows linearly with $F$ as shown in (2.40). At $F = F_c$, the semitrivial solution loses stability and the amplitude of the pitch-mode is limited when $F$ is increased further. The amplitude of the roll-mode however grows over time with increasing $F$. In other words, the pitch-mode becomes saturated and the excess energy “spills over” into the roll-mode. Figure 3.3 illustrates this behaviour, where black line represents the response amplitude of the primary system and the red line represents the amplitude of the secondary system. The blue line in the graph represents the forcing amplitude which is varied from zero to above critical levels. This graph is drawn based on the results of 100 numerical simulations in which the amplitudes of the oscillations of the primary- and secondary system were logged. In every simulation the excitation amplitude of the primary system was increased by a small factor. The parameters used in these simulations are given in the table displayed next to the graph. When $F \approx 16$, the primary system gets saturated and the excess energy "spills over" to the secondary system. The slope of the amplitude of the primary system decreases while the amplitude of the secondary system starts to grow. In accordance with the experiences of a ship which is advancing in a head or following sea the results show that, if the waves are big enough and at the right frequency, the vessel begins to roll violently.

![Figure 3.3: Saturation phenomenon in pitch-roll model.](image)

In the next discussion the phenomenon of the transfer of energy will be illu-
dicated with respect to tuning and detuning the resonance conditions. In contrast with the previous analysis during these experiments the forcing amplitude is kept constant and above the critical value for resonance to occur. However, in this case, the forcing frequency is first tuned for parametric roll resonance to occur, and after that detuned again. In order to analyse the transfer of energy between the primary system, pitch, and the secondary system, roll, the following energy functions are
extracted from the equations for potential- and kinetic energy given in Appendix C:

\[
E_{\text{pitch}} = \frac{1}{2} (m_1 + m_2) x_2^2, \quad (3.3)
\]

\[
E_{\text{roll}} = m_2 \left( \frac{1}{2} l^2 x_4^2 + gl (1 - \cos(x_3)) \right). \quad (3.4)
\]

These equations allow the calculation of the amount of energy in the primary- and secondary-system during each time step of the numerical simulation. These equations are composed on the basis of associating the different terms in the equations for potential- and kinetic-energy with the primary- or secondary-system. The terms which are a function of \(\dot{\theta}\), or \(x_2\) are collected in the function for the primary system, and terms which are a function of \(\phi\) or \(\dot{\phi}\) are collected in the function for the secondary system. Since this is model of a coupled system employing method is questionable, but for the purpose of evaluating the transfer of energy the results could be very interesting. Again, two terms are omitted here. The term containing the potential energy of the spring is considered to be generated by the forces acting on the hull resulting from the wave pressure, and the nonlinear coupling term which is a function of \(\dot{\theta}\), \(\phi\) and \(\dot{\phi}\), or \(x_2\), \(x_3\) and \(x_4\) respectively, is not associated with one of the particular systems. Together, these two omitted terms could account for approximately 10% to 30% of the system’s total energy.

With the state equation a numerical simulation can be conducted in Matlab, and with the given energy functions the energy transfer between the primary- and secondary system can be evaluated. In Figure 3.4, the results are shown for a numerical simulation with the values for the different parameters shown in the table above the graphs. In Appendix D the results are displayed in more detail. In both graphs the horizontal axis represents time, in the left graph the vertical axis represents pitch and roll and in the right graph the vertical axis represents energy. The yellow line in the left graph represents the oscillation in pitch and the red line represents the oscillation in roll. The black line in the right graph represents the oscillation of energy in the primary system and the yellow line represents the energy in the secondary system. During the simulation \(\eta\) is slowly increased from 3.9 to 4.0 until parametric resonance occurs and a large roll motion develops as a consequence of the transfer of energy from the primary system to the secondary system. After some time \(\eta\) is decreased to detune the parametric resonance and
the roll motion decays to smaller amplitudes. The detuning of the resonance by decreasing $\eta$, i.e. the excitation frequency or the wave encounter frequency in the case of a vessel, agrees with a change in speed or heading of a vessel.

### 3.3 Heave-pitch-roll model

The heave-pitch-roll model described in Section 2.7 is considered here. This system also displays a saturation phenomenon. When $\eta$ is near $\omega_{1,2}$ and $a < a_C$, the roll-mode is initially nonresponsive. The stable response of the system will be the semi-trivial solution (2.47). The amplitude of this solution grows linearly with $a$. At $a = a_c$ the semitrivial solution loses stability and a nontrivial solution appears. The amplitude of the roll-mode however grows over time with increasing $a$. In other words, the pitch-mode becomes saturated and the excess energy "spills over" into the roll-mode. Figure 3.5 illustrates this behaviour. In this graph the black line represents the amplitude of the primary systems’ response, the red line represents the amplitude of the secondary system and the blue line represents the forcing amplitude which is varied from zero to above critical levels. The amplitudes of the response of the different systems were logged during 100 numerical simulations in which the excitation amplitude was increased with a small increment for every new simulation. In the table next to the graph the parameters are shown which were used during the simulations. When $a \approx 0.028$, the primary system starts to get saturated and the excess energy "spills over" to the secondary system. The growth rate of the response amplitude of the primary system decreases, while the amplitude of the secondary systems’ response starts to increase. These results are in accordance with a ship which is advancing in a head or following sea, and, if the waves are big enough and at the right frequency, begins to roll violently.

The phenomenon of tuning and detuning the resonance condition by altering the forward speed or heading of a vessel is considered in the next assessment. In contrast with the previous analysis during these experiments the forcing amplitude is kept constant and above the critical value for resonance to occur. However, in this case the forcing frequency is first tuned for parametric roll resonance to occur, and after that detuned again. In order to analyse the energy transfer between the

![Figure 3.5: Saturation phenomenon in heave-pitch-roll model.](image)
primary system, heave and pitch, and the secondary system, roll, the following energy functions are compiled from the equations for potential- and kinetic energy given in Appendix E:

\[ E_{\text{heave+pitch}} = \frac{1}{2} (m_1 + m_2) x_2^2 + \frac{1}{2} (m_2 + m_4) x_4^2, \quad (3.5) \]

\[ E_{\text{roll}} = (m_3 + m_4) \left( \frac{1}{2} l_2^2 x_6^2 + gl (1 - \cos (x_5)) \right). \quad (3.6) \]

The equations are a result of collecting the terms associated to the primary- and secondary system in different equations. Terms which are a function of \( \dot{z}_1 \) or \( \dot{z}_2 \), or \( x_2 \) and \( x_4 \) respectively, are collected in the energy function for the primary system, and terms which are a function of \( \phi \) or \( \dot{\phi} \), or \( x_5 \) and \( x_6 \) respectively, are collected in the energy function for the secondary system. Since this is a model of a coupled system employing this method is questionable, but for the purpose of evaluating the transfer of energy the results could prove to be interesting. Also in this case some terms are omitted. Terms with respect to the potential energy of the two springs are considered to represent the forces acting on the hull generated by the wave pressure, and the nonlinear coupling terms which are a function of \( \dot{z}_1 \), \( \dot{z}_2 \), \( \phi \) and \( \dot{\phi} \), or respectively \( x_2 \), \( x_4 \), \( x_5 \) and \( x_6 \), are not associated with one system in particular. Together, these two omitted terms could account for approximately 10% to 30% of the system’s total energy.

With the state equation of the heave-pitch-roll model, given by (E.15) in Appendix E, a numerical simulation can be conducted in Matlab and with the given energy functions the energy transfer between the primary- and secondary system can be evaluated. In Figure 3.6 the results for a numerical simulation are displayed.

![Figure 3.6: Results of numerical simulation of heave-pitch-roll model.](image)

The values used for the different parameters are given in the table which is displayed alongside the graph. In Appendix F these results are displayed in greater detail. In both graphs the horizontal axis represents time, in the left graph the vertical axis represents heave and roll and in the right graph the vertical axis represents energy. The yellow line in the left graph represents the oscillation in heave and the red line represents the oscillation in roll. The black line in the right graph represents the energy in the primary system and the yellow line represents the energy...
in the secondary system. During the simulation parameter $\eta$ is slowly increased from 1.7 to 2.0 until parametric resonance occurs and roll motion with large amplitudes develops as a result of the transfer of energy from the primary system to the secondary system. After some time $\eta$ is decreased to detune the parametric resonance and the large roll motion decays to smaller amplitudes. The detuning of the parametric roll resonance by decreasing $\eta$, or the excitation frequency or wave encounter frequency in the case of a vessel, could comply with a change in speed or heading of a vessel.
Chapter 4

Conclusions and recommendations

4.1 Conclusions

The previous chapters described the dynamical analysis of three ship models and the evaluation of the transfer of energy during parametric roll resonance.

The conditions for parametric roll resonance to occur with the heave-roll model were identified and the transfer of energy was then evaluated. The saturation phenomenon is visualized, though a more dramatic phenomenon was expected from previous research [16]. The response amplitude of the primary system is expected to reach a maximum under certain circumstances after the excitation amplitude $a$ exceeds $a_{C}$, but this situation could not be reproduced exactly; only a decrease of the slope is obtained from simulations conducted by numerical integration.

The energy transfer between the primary- and secondary system during numerical simulations was evaluated and is clearly visible in the results, though the magnitude is uncomparable with a real ship due to the simplified model and the used parameters like the light mass for example. The relative values of the amplitude of the heave and pitch motions are slightly too large compared with real ships and the applied model cannot reproduce the complex phenomenon of ship-fluid interaction in great detail.

The pitch-roll models’ stability properties were analysed and thereby the conditions for parametric roll resonance to occur were obtained. The saturation phenomenon was reproduced and in this case shows a dramatic decrease in growth rate of the response amplitude of the primary system after $F$ exceeds $F_{c}$, while the secondary systems’ response amplitude starts to grow at this moment. Though, also in this simulation, a maximum for the response amplitude of the primary system was predicted to be reached at $F = F_{c}$, but the involved conditions could not be reproduced exactly.

The pitch motion shows amplitudes which are relatively large in comparison with a real vessel, but the magnitude of the roll motion is realistic. The magnitude of the involved energy is not representible for a real ship since the model is a simplified representation of a ship and the used masses for example are too light. The transfer of energy between the different modes is again clearly identifiable, but also this model cannot reproduce the complex features of ship-fluid interaction in great detail.
The stability properties of the solutions of the heave-pitch-roll-model were assessed and also the conditions for parametric roll resonance to occur are obtained. With numerical simulations the saturation phenomenon could be reproduced, though, for these simulation conditions, this model shows the fewest similarities with the expected results from previous research[16] in comparison with the former two models. Again the slope of the response amplitude of the primary system decreases after the critical value is reached, but it does not reach a maximum at that point. In contrast to this, the amplitude of the secondary systems’ response starts to grow dramatically which is as predicted.
The magnitude of the different modes during numerical simulation are realistic for a real vessel and the transfer of energy during parametric roll resonance is clearly visible. Again it should be noted that the magnitude of the amount of energy involved in the system is not representing that of a real ship since the model is a simplification and the used mass-settings are to light. Also this model has its limitations regarding the detailed reproduction of the complex ship-fluid interactions.

4.2 Recommendations

For future research projects it would be interesting to set up a mechanical model of one of the presented ship models and evaluate its behaviour. When the resonance conditions are met, a slight decrease in oscillation amplitude of the primary system is observed in the three assessed ship models, while the oscillation amplitude of the secondary system starts to grow rapidly. This is expected to result from the balance and transfer of energy. It is not certain if this is also experienced in real ships, and a mechanical model could answer this question. On the other hand the described phenomenon could also be investigated in a towing tank, but this procedure would be more expensive in the sense of time and resources.
Bibliography


Appendix A

Equations of motion for heave-roll model

The generalized coordinates for this model are:

\[ q = \begin{bmatrix} z \\ \varphi \end{bmatrix}. \]  \hspace{1cm} (A.1)

The coordinates of mass \( m_1 \) are:

\[ r_{m_1} = \begin{bmatrix} z \\ 0 \end{bmatrix}. \]  \hspace{1cm} (A.2)

The velocities of mass \( m_1 \) are then defined as:

\[ \dot{r}_{m_1} = \begin{bmatrix} \dot{z} \\ 0 \end{bmatrix}. \]  \hspace{1cm} (A.3)

The coordinates of mass \( m_2 \) are:

\[ r_{m_2} = \begin{bmatrix} z - l\cos(\varphi) \\ l\sin(\varphi) \end{bmatrix}. \]  \hspace{1cm} (A.4)

The velocities of mass \( m_2 \) are then defined as:

\[ \dot{r}_{m_2} = \begin{bmatrix} \dot{z} + \dot{\varphi}l\sin(\varphi) \\ \dot{\varphi}l\cos(\varphi) \end{bmatrix}. \]  \hspace{1cm} (A.5)

The kinetic energy of the system is now given by:

\[ T = \frac{1}{2}m_1 \dot{z}^2 + \frac{1}{2}m_2 (\dot{z} + l\dot{\varphi}\sin(\varphi))^2 + \frac{1}{2}m_2 (l\dot{\varphi}\cos(\varphi))^2. \]  \hspace{1cm} (A.6)

The potential energy of the system reads:

\[ V = m_2gl(1 - \cos(\varphi)) + \frac{1}{2}kz^2. \]  \hspace{1cm} (A.7)

The Lagrange’s equations of motion of this system are:

\[ (m_1 + m_2) (\ddot{z} - \alpha \omega^2 \cos(\omega t)) + b\dot{z} + kz + ml (\ddot{\varphi}\sin(\varphi) + \dot{\varphi}^2 \cos(\varphi)) = 0, \]

\[ ml^2 \ddot{\varphi} + c\dot{\varphi} + mgl\sin(\varphi) + ml (\ddot{z} - \alpha \omega^2 \cos(\omega t) \sin(\varphi)) = 0. \]  \hspace{1cm} (A.8)
By means of the new time variable \( \tau = (g/l)^{\frac{1}{2}} t \), the equations of motion can be written in dimensionless form [23]:

\[
\ddot{w} + \kappa \dot{w} + q^2 w + \mu \left( \ddot{\phi} \sin(\phi) + \dot{\phi}^2 \cos(\phi) \right) = \alpha \eta^2 \cos(\eta \tau), \\
\ddot{\phi} + \kappa_0 \dot{\phi} + \sin(\phi) + \left( \dot{w} - \alpha \eta^2 \cos(\eta \tau) \right) \sin(\phi) = 0,
\]

(A.9)

where \( w = \frac{z}{l} \), \( \omega_0 = \sqrt{\frac{g}{l}} \), \( \kappa = \frac{b \omega_0}{m_1 + m_2} \), \( q^2 = \frac{\kappa \omega_0^2}{m_1 + m_2} \), \( \mu = \frac{m_2}{m_1 + m_2} \kappa_0 = \left( \frac{\omega_0}{c_{\omega_0}} \right) ml^2 \), \( \eta = \frac{\omega}{\omega_0} \) and \( a = \frac{\alpha}{l} \).

With:

\[
x_1 = w, \\
x_2 = \dot{w}, \\
x_3 = \phi, \\
x_4 = \dot{\phi},
\]

(A.10)

the state equation of system (A.9) becomes:

\[
\dot{x}_1 = x_2, \\
\dot{x}_2 = \frac{\alpha \eta^2 \cos(\eta \tau) - \kappa x_2 - q^2 x_1 + x_4^2 \cos(x_3)}{1 - \mu \sin^2(x_3)} - \\
\mu \left( -\kappa_0 x_4 - \sin(x_3) + \alpha \eta^2 \cos(\eta \tau) \sin(x_3) \right) \sin(x_3), \\
\dot{x}_3 = x_4, \\
\dot{x}_4 = \frac{-\kappa_0 x_4 - \left( 1 + \left( \alpha \eta^2 - \alpha \eta^2 \cos(\eta \tau) \right) \sin(x_3) \right)}{1 - \mu \sin^2(x_3)} - \\
\frac{\left( \kappa x_2 - q^2 x_1 - \mu x_4^2 \cos(x_3) \right) \sin(x_3)}{1 - \mu \sin^2(x_3)}. 
\]

(A.11)
Appendix B

Results of numerical simulation of heave-roll model

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<th>q</th>
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<th>m₂ [kg]</th>
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Figure B.1: Parameters of heave-roll model used during simulation.

The graphs with the simulation results of the heave-roll model are displayed on the next pages.

Figure B.2: Results of numerical simulation of heave-roll model.
Figure B.3: Results of numerical simulation of heave-roll model in greater detail.

Figure B.4: Energy transfer of heave-roll model during numerical simulation.
Figure B.5: Energy transfer of heave-roll model during numerical simulation in greater detail.
Appendix C

Equations of motion for pitch-roll model

The generalized coordinates for the mechanism displayed in Figure 2.10 are:

\[ q = \begin{bmatrix} \theta \\ \varphi \end{bmatrix}. \]  \hspace{1cm} (C.1)

The coordinates of the center of the rigid block are:

\[ (x_1)_{m_1} = x_{10} + \theta, \]
\[ (x_2)_{m_1} = x_{20}. \]  \hspace{1cm} (C.2)

Herein are \( x_{10} \) and \( x_{20} \) time-independent reference positions. The coordinates of the mass \( m_2 \) are:

\[ (x_1)_{m_2} = x_{10} + \theta + l\sin(\varphi), \]
\[ (x_2)_{m_2} = h - l\cos(\varphi). \]  \hspace{1cm} (C.3)

Herein is \( h \) a time-independent reference position of the hinge. The velocities of \( m_1 \) and \( m_2 \) are:

\[ (\dot{x}_1)_{m_1} = \dot{\theta}, \]
\[ (\dot{x}_2)_{m_1} = 0, \]
\[ (\dot{x}_1)_{m_2} = \dot{\theta} + l\dot{\varphi}\cos(\varphi), \]
\[ (\dot{x}_2)_{m_2} = l\dot{\varphi}\sin(\varphi). \]  \hspace{1cm} (C.4)

The kinetic energy of the system now equals:

\[ T = \frac{1}{2} (m_1 + m_2) \dot{\theta}^2 + m_2 \dot{\theta}\dot{\varphi}\cos(\varphi) + \frac{1}{2} m_2 l^2 \dot{\varphi}^2. \]  \hspace{1cm} (C.5)

The potential energy of the system is:

\[ V = \frac{1}{2} k\theta^2 + m_2 g [h - l\cos(\varphi)]. \]  \hspace{1cm} (C.6)
With:

\[
\begin{align*}
    x_1 &= \theta, \\
    x_2 &= \dot{\theta}, \\
    x_3 &= \varphi, \\
    x_4 &= \dot{\varphi},
\end{align*}
\] (C.7)

the state equation of system (2.37) becomes:

\[
\begin{align*}
    \dot{x}_1 &= x_2, \\
    \dot{x}_2 &= F\eta^2 \cos(\eta \tau) - \mu_1 x_2 - \theta - \alpha_1 \varphi^2, \\
    \dot{x}_3 &= x_4, \\
    \dot{x}_4 &= -\mu_2 \dot{\varphi} - q^2 \varphi - \alpha_2 \theta \varphi.
\end{align*}
\] (C.8)
Appendix D

Results of numerical simulation of pitch-roll model

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</tbody>
</table>

Figure D.1: Parameters of pitch-roll model used during simulation.

On the next pages the results of a numerical simulation with the pitch-roll model are displayed.

Figure D.2: Results of numerical simulation of heave-roll model.
Figure D.3: Results of numerical simulation of pitch-roll model in greater detail.

Figure D.4: Energy transfer of pitch-roll model during numerical simulation.
Figure D.5: Energy transfer of pitch-roll model during numerical simulation in greater detail.
Appendix E

Equations of motion for heave-pitch-roll model

The generalized coordinates for this model are:

\[ q = \begin{bmatrix} z_1 \\ z_2 \\ \varphi \end{bmatrix}. \]  
(E.1)

The coordinates of mass \( m_1 \) are:

\[ r_{m_1} = \begin{bmatrix} z_1 \\ 0 \end{bmatrix}. \]  
(E.2)

The velocities of mass \( m_1 \) are then defined as:

\[ \dot{r}_{m_1} = \begin{bmatrix} \dot{z}_1 \\ 0 \end{bmatrix}. \]  
(E.3)

The coordinates of mass \( m_3 \) are:

\[ r_{m_3} = \begin{bmatrix} z_1 - l\cos(\varphi) \\ l\sin(\varphi) \end{bmatrix}. \]  
(E.4)

The velocities of mass \( m_3 \) are then defined as:

\[ \dot{r}_{m_3} = \begin{bmatrix} \dot{z}_1 + \dot{\varphi}l\sin(\varphi) \\ \dot{\varphi}l\cos(\varphi) \end{bmatrix}. \]  
(E.5)

The coordinates of mass \( m_2 \) are:

\[ r_{m_2} = \begin{bmatrix} z_2 \\ 0 \end{bmatrix}. \]  
(E.6)

The velocities of mass \( m_2 \) are then defined as:

\[ \dot{r}_{m_2} = \begin{bmatrix} \dot{z}_2 \\ 0 \end{bmatrix}. \]  
(E.7)
The coordinates of mass $m_4$ are:

$$r_{m_4} = \begin{bmatrix} z_2 - l \cos (\varphi) \\ l \sin (\varphi) \end{bmatrix}.$$ \hspace{1cm} (E.8)

The velocities of mass $m_4$ are then defined as:

$$\dot{r}_{m_4} = \begin{bmatrix} \dot{z}_2 + \dot{\varphi} l \sin (\varphi) \\ \dot{\varphi} l \cos (\varphi) \end{bmatrix}.$$ \hspace{1cm} (E.9)

The kinetic energy of this system is:

$$T = \frac{1}{2} m_1 \dot{z}_1^2 + \frac{1}{2} m_3 (\dot{z}_1 + l \dot{\varphi} \sin (\varphi))^2 + \frac{1}{2} m_2 \dot{z}_2^2 + \frac{1}{2} m_4 (\dot{z}_2 + l \dot{\varphi} \sin (\varphi))^2 + \frac{1}{2} (m_3 + m_4) (l \dot{\varphi} \cos (\varphi))^2.$$ \hspace{1cm} (E.10)

The potential energy of the system equals:

$$V = (m_3 + m_4) gl (1 - \cos (\varphi)) + \frac{1}{2} k_1 z_1^2 + \frac{1}{2} k_2 z_2^2.$$ \hspace{1cm} (E.11)

The Lagrange’s equations of motion of this system are:

$$(m_1 + m_3) (\ddot{z}_1 - \alpha \omega^2 \cos (\omega t)) + b_1 \dot{z}_1 + b_2 z_2 + m_4 \dot{\varphi} \sin (\varphi) + \dot{\varphi}^2 \cos (\varphi) = 0,$$

$$(m_2 + m_4) (\ddot{z}_2 - \alpha \omega^2 \cos (\omega t)) + b_2 \dot{z}_2 + k_2 z_2 + m_4 \dot{\varphi} \sin (\varphi) + \dot{\varphi}^2 \cos (\varphi) = 0,$$

$$(m_3 + m_4) l^2 \ddot{\varphi} + c \dot{\varphi} + (m_3 + m_4) gl \sin (\varphi) + (m_3 + m_4) l (\ddot{z}_1 - \alpha \omega^2 \cos (\omega t) + \ddot{z}_2 - \alpha \omega^2 \cos (\omega t - \psi)) \sin (\varphi) = 0.$$ \hspace{1cm} (E.12)

By means of the time variable $\tau = (g/l)^{1/2} t$, the equations of motion can be written in the following dimensionless form [23]:

$$\tilde{w}_1 + \kappa_1 \tilde{w}_1 + q_1 \tilde{w}_1 + \mu_1 (\tilde{\varphi} \sin \varphi + \dot{\varphi}^2 \cos \varphi) = a \eta^2 \cos (\eta \tau),$$

$$\tilde{w}_2 + \kappa_2 \tilde{w}_2 + q_2 \tilde{w}_2 + \mu_2 (\tilde{\varphi} \sin \varphi + \dot{\varphi}^2 \cos \varphi) = a \eta^2 \cos (\eta \tau - \psi),$$

$$\ddot{\varphi} + \kappa_0 \dot{\varphi} + \sin \varphi + \frac{1}{2} \left[ \ddot{\varphi} - a \eta^2 \cos \eta \tau + \ddot{\varphi} - a \eta^2 \cos (\eta \tau - \psi) \right] \sin \varphi = 0.$$ \hspace{1cm} (E.13)

where $w_j = \frac{z_j}{r}$, $\omega_0 = \sqrt{g/l}$, $\kappa_j = \left( \frac{b_j}{\omega_0} \right) (m_j + m_{j+2})$, $q_j^2 = \left( \frac{a_j}{\omega_0} \right) (m_j + m_{j+2})$, $\mu_j = \frac{m_{j+2}}{m_j + m_{j+2}}$, $\kappa_0 = \left( \frac{a_0}{\omega_0} \right) m_{j+2} \ell^2$, $\eta = \frac{w_j}{\omega_0}$, $a = \frac{\alpha}{\ell}$ and $j = 1, 2$.

With:

$$x_1 = w_1,$$

$$x_2 = \tilde{w}_1,$$

$$x_3 = w_2,$$

$$x_4 = \tilde{w}_2,$$

$$x_5 = \varphi,$$

$$x_6 = \dot{\varphi}.$$ \hspace{1cm} (E.14)
the state equation of system (E.13) becomes:

\[ \begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= a \eta^2 \cos(\eta \tau) - \kappa_1 x_2 - q_1^2 x_1 + \\
&\quad \mu_1 \sin(x_5) \left( \frac{\kappa_0 x_6 + \sin(x_5) + \frac{1}{6} \sin(x_5) \left( a \eta^2 \cos(\eta \tau) + \kappa_1 x_2 \right)}{1 - \frac{1}{3} (\mu_1 + \mu_2) \sin^2(x_5)} \right) - \\
&\quad \mu_1 \sin(x_5) \left( \frac{\frac{1}{6} \sin(x_5) \left( q_1^2 x_1 + \mu_1 x_6^2 \cos(x_5) + a \eta^2 \cos(\eta \tau - \psi) \right)}{1 - \frac{1}{3} (\mu_1 + \mu_2) \sin^2(x_5)} \right) - \\
&\quad \mu_1 \sin(x_5) \left( \frac{\frac{1}{6} \sin(x_5) \left( \kappa_2 x_4 + q_2^2 x_3 + \mu_2 x_6^2 \cos(x_5) \right)}{1 - \frac{1}{3} (\mu_1 + \mu_2) \sin^2(x_5)} \right) + \\
&\quad \mu_1 \sin(x_5) \left( \frac{\eta^2 \cos(\eta \tau - \psi) \sin(x_5)}{1 - \frac{1}{3} (\mu_1 + \mu_2) \sin^2(x_5)} - \kappa_2 \sin(x_5) \right), \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= a \eta^2 \cos(\eta \tau - \psi) - \kappa_2 x_4 - q_2^2 x_3 + \\
&\quad \mu_2 \sin(x_5) \left( \frac{\kappa_0 x_6 + \sin(x_5) + \frac{1}{6} \sin(x_5) \left( a \eta^2 \cos(\eta \tau) + \kappa_1 x_2 \right)}{1 - \frac{1}{3} (\mu_1 + \mu_2) \sin^2(x_5)} \right) - \\
&\quad \mu_2 \sin(x_5) \left( \frac{\frac{1}{6} \sin(x_5) \left( q_1^2 x_1 + \mu_1 x_6^2 \cos(x_5) + a \eta^2 \cos(\eta \tau - \psi) \right)}{1 - \frac{1}{3} (\mu_1 + \mu_2) \sin^2(x_5)} \right) - \\
&\quad \mu_2 \sin(x_5) \left( \frac{\frac{1}{6} \sin(x_5) \left( \kappa_2 x_4 + q_2^2 x_3 + \mu_2 x_6^2 \cos(x_5) \right)}{1 - \frac{1}{3} (\mu_1 + \mu_2) \sin^2(x_5)} \right) + \\
&\quad \mu_2 \sin(x_5) \left( \frac{\eta^2 \cos(\eta \tau - \psi) \sin(x_5)}{1 - \frac{1}{3} (\mu_1 + \mu_2) \sin^2(x_5)} - \kappa_2 \sin(x_5) \right), \\
\dot{x}_5 &= x_6, \\
\dot{x}_6 &= -\kappa_0 x_6 - \sin(x_5) - \frac{1}{6} \sin(x_5) \left( a \eta^2 \cos(\eta \tau) - \kappa_1 x_2 - q_1^2 x_1 \right) - \\
&\quad \frac{1}{6} \sin(x_5) \left( -\mu_1 x_6^2 \cos(x_5) - a \eta^2 \cos(\eta \tau - \psi) - \kappa_2 x_4 - q_2^2 x_3 \right) - \\
&\quad \frac{1}{6} \sin(x_5) \left( -\mu_2 x_6^2 \cos(x_5) - \eta^2 \cos(\eta \tau - \psi) \right) - \\
&\quad \frac{1}{6} \sin(x_5) \left( -\mu_2 x_6^2 \cos(x_5) - \eta^2 \cos(\eta \tau - \psi) \right) \left( \frac{1}{1 - \frac{1}{3} (\mu_1 + \mu_2) \sin^2(x_5)} \right). \\
\end{align*} \] (E.15)
Appendix F

Results of numerical simulation of heave-pitch-roll model

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<th>$q_{10}$</th>
<th>$m_1$ [kg]</th>
<th>$m_2$ [kg]</th>
<th>$m_3$ [kg]</th>
<th>$m_4$ [kg]</th>
<th>$I_1$ [m]</th>
<th>$I_2$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.1</td>
<td>1.7</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>9.6</td>
</tr>
</tbody>
</table>

Figure F.1: Parameters of heave-pitch-roll model used during simulation.

The results of the numerical simulations with the heave-pitch-roll model are shown on the next pages.

Figure F.2: Results of numerical simulation of heave-pitch-roll model.

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Figure F.3: Results of numerical simulation of heave-pitch-roll model in greater detail.

Figure F.4: Energy transfer during parametric roll resonance of heave-pitch-roll model.
Figure F.5: Energy transfer of heave-roll-pitch model during parametric roll resonance in greater detail.