Synchronization of Diffusively Coupled Electronic Hindmarsh-Rose Oscillators

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Abstract

In this chapter we present results on synchronization of diffusively coupled Hindmarsh-Rose (HR) electronic oscillators. These electronic oscillators are analog electrical circuits which integrate the differential equations of the Hindmarsh-Rose model. An experimental setup consisting of four chaotic Hindmarsh-Rose oscillators, is used to evaluate the existence and stability of partially or fully synchronized states.

13.1 Introduction

Synchronous behavior of systems is witnessed in a vast number of research area’s. Beautiful examples are, for instance, the simultaneous flashing of male fireflies on banks along rivers in Malaysia, Thailand and New Guinea [Strogatz and Stewart, 1993], and the synchronous release of action potentials in parts of the mammalian brain [Gray, 1994]. A large number of examples of synchronization in nature can be found in [Pikovsky et al., 2003]. Synchronization can also be found in robotics, usually referred to as coordination [Rodriguez-Angeles and Nijmeijer, 2001], and it is potentially of interest in the field of (secure) communication, see for instance [Pecora
and Carroll, 1990; Huijberts et al., 1998].

Most of the studies on synchronization in networks of coupled systems deal with analysis supported by simulations. Significantly less attention is given to validate synchronization in an experiment setup. In this paper we present synchronization of coupled Hindmarsh-Rose electronic oscillators. The Hindmarsh-Rose model [Hindmarsh and Rose, 1984] is a well-known model in the field of neuroscience that provides a description of the action potential generation in neuronal cells. This model consists of three coupled nonlinear differential equations and is capable, as function of specific parameters, of producing both simple and complex oscillatory motion. Here, an experimental setup consisting of four circuits, operating in a chaotic regime, is used to investigate the existence and stability of synchronized states.

The Hindmarsh-Rose oscillators in the experimental setup are diffusively coupled; that is the systems are mutually coupled using linear functions of the outputs of the systems. Using a semipassivity based approach [Pogromsky et al., 2002], we derive conditions that guarantee the existence of synchronized regimes. These regimes might correspond to the fully synchronized state, i.e. all systems perform an identical motion, as well as to partial synchronization where only some systems do synchronize.

This chapter is organized as follows. In section 13.2 the mathematical notations are being introduced and we present the notions of semipassivity and convergent systems. In section 13.3 we present a theoretical passivity-based framework, introduced in [Pogromsky, 1998], that provides conditions under which the coupled oscillators synchronize. In addition, we show that the Hindmarsh-Rose systems satisfy the assumptions of this framework. Next, in section 13.4 the experimental setup is discussed and in section 13.5 we show that the coupled electronic Hindmarsh-Rose systems synchronize. Finally, in section 13.6 conclusions are drawn.

13.2 Preliminaries

Throughout this paper we use the following notations. The Euclidian norm in $\mathbb{R}^n$ is denoted by $\| \cdot \|$, $\|x\|^2 = x^\top x$ where the symbol $^\top$ stand for transposition. The symbol $I_n$ defines the $n \times n$ identity matrix and the notation $\text{col} (x_1, \ldots, x_n)$ stands for the column vector containing the elements $x_1, \ldots, x_n$. A function $V : \mathbb{R}^n \to \mathbb{R}^+$ is called positive definite if $V(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $V(0) = 0$. It is radially unbounded if $V(x) \to \infty$ if
∥x∥ → ∞. If the quadratic form \( x^\top P x \) with a symmetric matrix \( P = P^\top \) is positive definite, then the matrix \( P \) is positive definite, denoted as \( P > 0 \).

The notation \( A \otimes B \) stands for the Kronecker product of the matrices \( A \) and \( B \).

We define (practical) synchronization and (practical) partial synchronization as follows:

**Definition 13.1 (synchronization).** Consider \( k \) interconnected dynamical systems with state variables \( x_i \in \mathbb{R}^n, i = 1, 2, \ldots, k \). The coupled systems are called

- synchronized if \( \lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0 \) for all \( i, j = 1, 2, \ldots, k \);
- practically synchronized if \( \limsup_{t \to \infty} \|x_i(t) - x_j(t)\| = \delta \) for all \( i, j = 1, 2, \ldots, k \) and some fixed, sufficiently small \( \delta > 0 \);
- partially synchronized if \( \lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0 \) for some \( i, j = 1, 2, \ldots, k \);
- practically partially synchronized if \( \limsup_{t \to \infty} \|x_i(t) - x_j(t)\| = \delta \) for some \( i, j = 1, 2, \ldots, k \) and some fixed, sufficiently small \( \delta > 0 \).

Let us, in addition, present the notions of semipassivity and convergent systems.

**Definition 13.2 (semipassivity).** [Pogromsky and Nijmeijer, 2001]

Consider the following system:

\[
\begin{align*}
\dot{x} &= f(x) + Bu \\
y &= Cx
\end{align*}
\]  

(13.1)

where state \( x \in \mathbb{R}^n \), input \( u \in \mathbb{R}^m \), output \( y \in \mathbb{R}^m \), vector field \( f : \mathbb{R}^n \to \mathbb{R}^n \) and matrices \( B \) and \( C \) of appropriate dimensions. Let \( V : \mathbb{R}^n \to \mathbb{R}_+ \), \( V(0) = 0 \) be a differentiable nonnegative (storage) function, then the system (13.1) is called semipassive if the following inequality is satisfied:

\[
\dot{V} \leq y^\top u - H(x)
\]  

(13.2)

where \( H : \mathbb{R}^n \to \mathbb{R} \) is nonnegative outside some ball

\[
\exists \rho > 0, \quad \forall \|x\| \geq \rho \Rightarrow H(x) \geq g(\|x\|)
\]

for some continuous nonnegative function \( g(\cdot) \) defined for \( \|x\| \geq \rho \). If the inequality (13.2) is strict, the system (13.1) is called strictly semipassive.

The most useful property of semipassive systems is that being linearly interconnected, the solutions of all systems in the network exist for all \( t \geq 0 \) and are ultimately bounded [Pogromsky, 1998].
Consider the following system:

$$\dot{z} = q(z, w)$$  \hspace{1cm} (13.3)

where $z(t) \in \mathbb{R}^s$, $w(t) \in \mathcal{D}$, $\mathcal{D}$ is some compact subset of $\mathbb{R}^p$, continuous function $w: \mathbb{R}_+ \rightarrow \mathcal{D}$ and the vector field $q: \mathbb{R}^s \times \mathcal{D} \rightarrow \mathbb{R}^s$.

**Definition 13.3 (convergent systems).** [Demidovich, 1967; Pavlov et al., 2006] The system (13.3) is said to be convergent if for any $w(t)$:

1. all solutions $z(t)$ are well-defined for all $t \in [t_0, +\infty)$ and all initial conditions $t_0 \in \mathbb{R}$, $z(t_0) \in \mathbb{R}^s$;
2. there exists a unique globally asymptotically stable solution $z_w(t)$ defined and bounded for all $t \in (-\infty, +\infty)$, i.e. for any solution $z(t)$ it follows that

$$\lim_{t \rightarrow \infty} \|z(t) - z_w(t)\| = 0.$$

According to Demidovich [Demidovich, 1967], there exists a simple sufficient condition to determine if the system (13.3) is convergent.

**Lemma 13.1.** [Demidovich, 1967; Pavlov et al., 2006] If there exists a matrix $P = P^\top > 0$ such that the eigenvalues $\lambda_i(Q)$ of the symmetric matrix

$$Q(z, w) = \frac{1}{2} \left( P \frac{\partial q}{\partial z}(z, w) + \frac{\partial q}{\partial z}(z, w) P^\top \right)$$  \hspace{1cm} (13.4)

are negative and separated away from the imaginary axis for all $z \in \mathbb{R}^s$, $w \in \mathcal{D}$, then the system (13.3) is convergent.

### 13.3 Synchronization of Diffusively Coupled Hindmarsh-Rose Oscillators

In this section conditions are posed that guarantee (partial) synchronization in a network of coupled Hindmarsh-Rose oscillators. First the semipassivity based framework as described in [Pogromsky et al., 2002] is presented, and next we show that the Hindmarsh-Rose oscillators satisfy the assumptions of this framework.

Consider the $k$ systems of the following form

$$\dot{x}_i = f(x_i) + Bu_i$$
$$y_i = Cx_i$$  \hspace{1cm} (13.5)
where $i = 1, 2, \ldots, k$ denotes the number of each system in the network, $x_i \in \mathbb{R}^n$ the state, $u_i \in \mathbb{R}^m$ the input and $y_i \in \mathbb{R}^m$ the output of the $i$th system, smooth vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and matrices $B$ and $C$ of appropriate dimensions. Let, in addition, the matrix $CB$ be positive definite and nonsingular.

The $k$ dynamical systems (13.5) are coupled via diffusive coupling, i.e. mutual interconnection through linear output coupling of the form

$$u_i = -\gamma_{1i}(y_i - y_1) - \gamma_{12}(y_i - y_2) - \cdots - \gamma_{1k}(y_k - y_1)$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$ denotes the strength of the interconnection between the systems $i$ and $j$.

Defining the $k \times k$ coupling matrix as

$$\Gamma = \begin{bmatrix}
\sum_{j=2}^{k} \gamma_{1j} & -\gamma_{12} & \cdots & -\gamma_{1k} \\
-\gamma_{21} & \sum_{i=1, i \neq 2}^{k} \gamma_{2i} & \cdots & -\gamma_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
-\gamma_{k1} & -\gamma_{k2} & \cdots & \sum_{i=1}^{k-1} \gamma_{ki}
\end{bmatrix}$$

the diffuse coupling functions (13.6) can be written as

$$\underline{u} = -\Gamma \underline{y}$$

where $\underline{u} = \text{col}(u_1, \ldots, u_k)$, $\underline{y} = \text{col}(y_1, \ldots, y_k)$. Since $\Gamma = \Gamma^T$ all its eigenvalues are real. Moreover, applying Gerschgorin’s theorem about the localization of the eigenvalues, it is easy to verify that $\Gamma$ is positive semidefinite.

A network might possess certain symmetries. In particular, the network may contain repeating patterns. Hence, a permutation of some elements in the network, with respect to the interconnections, will leave the network unchanged. The mathematical representation of the permutation of the elements is a permutation matrix $\Pi \in \mathbb{R}^{k \times k}$. The matrix $\Pi$ defines a symmetry for the network if $\Gamma$ and $\Pi$ commute, i.e. $\Pi\Gamma = \Gamma\Pi$. Moreover, given a permutation matrix $\Pi$ that commutes with $\Gamma$, the set $\ker(I_{kn} - \Pi \otimes I_n)$ defines a linear invariant manifold for the closed loop systems (13.5) and (13.7). To be precise, the set $\ker(I_{kn} - \Pi \otimes I_n)$ describes a set of linear equations of the form $x_i - x_j = 0$ for some $i$ and $j$. Hence, we want to guarantee asymptotic stability of such a set. Therefore, introduce a linear change of coordinates $x_i \mapsto (z_i, y_i)$. Under the assumption that $CB$ is nonsingular, the systems (13.5) can be written after the coordinate transformation in the normal form:

$$\begin{cases}
\dot{z}_i = q(z_i, y_i) \\
\dot{y}_i = a(z_i, y_i) + CBu_i
\end{cases}$$

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where $z_i \in \mathbb{R}^{n-m}$ and smooth vector fields $q : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^{n-m}, \ a : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$. A sufficient condition for asymptotic stability of the set $\ker (I_kn - \Pi \otimes I_n)$ is given in the following theorem:

**Theorem 13.1.** [Pogromsky et al., 2002] Let $\lambda'$ be the smallest nonzero eigenvalue of $\Gamma$ under the restriction that the eigenvectors are taken from the set $\text{range} (I_kn - \Pi)$. Assume that:

1. the free system (13.8) is strictly semipassive with respect to input $u_i$ and output $y_i$ with a radially unbounded storage function;
2. there exists a matrix $P = P^T > 0$ such that the conditions of Lemma 13.1 are satisfied for $q$ as defined in (13.8).

Then for all positive semidefinite matrices $\Gamma$ all solutions of the diffusive network (13.8) and (13.7) are ultimately bounded and there exists a positive number $\bar{\lambda}$ such that if $\lambda' \geq \bar{\lambda}$ the set $\ker (I_kn - \Pi \otimes I_n)$ contains a globally asymptotically stable subset.

A network of Hindmarsh-Rose oscillators is given by the following set of equations:

$$
\begin{aligned}
\dot{y}_i &= -a_1y_i^3 + a_2y_i + a_3 + a_4z_{1,i} - a_5z_{2,i} + v + u_i \\
\dot{z}_{1,i} &= -b_1 - b_2y_i^2 - b_3y_i - b_4z_{1,i} \\
\dot{z}_{2,i} &= c_1(c_2(y_i + c_3) - z_{2,i})
\end{aligned}
$$

where $\tau := \frac{\textrm{d}}{\textrm{d}t}$, $\tau = 1000t$, $i = 1, 2, \ldots, k$ denotes the number of each oscillator in the network, $y_i$ represents the membrane potential, which can be regarded as the natural output of a neuron, $z_{1,i}$ is an internal recovery variable and $z_{2,i}$ is a slow internal recovery variable and input $u_i$. The constant $v$ represents an external applied stimulus. Parameters $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, c_1, c_2, c_3$ are all positive constants.

**Lemma 13.2.** Each free Hindmarsh-Rose system is strictly semipassive with respect to the input $u_i$ and the output $y_i$ with a radially unbounded storage function.

**Proof.** Following [Oud and Tyukin, 2004], consider the following storage function $V : \mathbb{R}^3 \to \mathbb{R}_+$:

$$
V(y_i, z_{1,i}, z_{2,i}) = \frac{1}{2} \left( y_i^2 + \mu z_{1,i}^2 + \frac{a_5}{c_1c_2}z_{2,i}^2 \right)
$$

with constant $\mu > 0$. Then

$$
\dot{V}(y_i, z_{1,i}, z_{2,i}) = y_iu_i - H(y_i, z_{1,i}, z_{2,i})
$$
where constants $\gamma_1, \gamma_2 \in (0, 1)$, $0 < \mu < \frac{4a_1(1-\gamma_1)}{b_2^2}$ and

$$H(y_i, z_{1,i}, z_{2,i}) = a_1 \gamma_1 y_i^4 - \left( a_2 + \frac{(a_4 + b_3 \mu)^2}{4b_4(1-\gamma_2)} \right) y_i^2 - (a_3 + v) y_i$$

$$+ \left( \mu \gamma_2 - \frac{\mu^2 b_2^2}{4a_1(1-\gamma_1)} \right) z_{1,i}^2 + \mu b_1 z_{1,i}$$

$$+ \frac{a_5}{c_2} z_{2,i} - a_5 c_3 z_{2,i}$$

$$+ a_1 (1-\gamma_1) \left( y_i^2 + \frac{\mu b_2}{2a_1(1-\gamma_1) z_{1,i}} \right)^2$$

$$+ \mu (1-\gamma_2) b_4 \left( z_{1,i} - \frac{a_4 - \mu b_3}{2b_4(1-\gamma_2)} y_i \right)^2.$$ 

Clearly, $H(\cdot) > 0$ for large $\|\text{col}(y_i, z_{1,i}, z_{2,i})\|$, i.e. each Hindmarsh-Rose oscillator satisfies the semipassivity condition (13.2). □

Lemma 13.3. The system

$$\begin{cases}
\dot{z}_{1,i} &= -b_1 - b_2 y_i^2 - b_3 y_i - b_4 z_{1,i} \\
\dot{z}_{2,i} &= c_1 (c_2 (y_i + c_3) - z_{2,i})
\end{cases} \quad (13.11)$$

is convergent.

Proof. Set $P = I_2$, then the matrix $Q$ as defined in (13.4) is given by

$$Q = \begin{bmatrix} -b_4 & 0 \\ 0 & -c_1 \end{bmatrix}.$$ 

Since $b_4, c_1 > 0$, it follows directly that the condition of Lemma 13.1 is satisfied, i.e. the system (13.11) is convergent. □

Lemmas 13.2 and 13.3 show that assumptions A1 and A2 of Theorem 13.1 hold and therefore we ensure that for sufficiently strong coupling the Hindmarsh-Rose systems (13.9) in the network will (partially) synchronize.

13.4 Experimental Setup

We have realized four analog electronic equivalents of the Hindmarsh-Rose equations (13.9), partially based on the implementation as presented in [Lee et al., 2004], with the following set of nominal parameters:

- $a_1 = 1$, $a_2 = 3$, $a_3 = 2$, $a_4 = 5$, $a_5 = 1$,
- $v = 3.25$, $b_1 = 0.8$, $b_2 = 1$, $b_3 = 2$, $b_4 = 1$,
- $c_1 = 0.005$, $c_2 = 4$, $c_3 = 2.618$. 

Each electronic Hindmarsh-Rose system consists of three integrator circuits, which integrate the Hindmarsh-Rose equations, and a multiplier circuit, build using AD633j voltage multipliers, that generates the squared and cubic terms in the Hindmarsh-Rose equations. Figure 13.1 shows a single electronic Hindmarsh-Rose circuit, and Figure 13.2 shows measured chaotic signals from such an electronic oscillator. In particular, in this figure the time series of the states $y, z_1, z_2$ are depicted and (a part of) the chaotic attractor in the phase-space is shown.

![The Hindmarsh-Rose electronic circuit.](image)

There are slight differences between the measured signals and the signals that can obtained through numerical integration of the equations (13.9). This mismatch is due to tolerances of the used components, i.e. the parameters of each circuit differ a little from the nominal ones. This implies that there are small differences between the individual circuits as well. Hence, synchronization in the sense that $x_i = x_j$, where $x_i = [y_i, z_{1,i}, z_{2,i}]^T$, is not possible. Therefore, we can only expect that the systems will \textit{practically (partially) synchronize} (as defined in Definition 13.1). Hence, we say that two systems $i$ and $j$ practically synchronize whenever $\limsup_{t \to \infty} \|x_i(t) - x_j(t)\| \leq 0.5 \ [V]$.

\textbf{Remark 13.1.} Although the value $\delta = 0.5 \ [V]$ seems rather high, one has to realize that due to the spiking behavior of the signals (see Figure 13.2), a small mismatch induces a relatively large error.
In order to define the connections between the Hindmarsh-Rose electronic systems, a synchronization interface is developed that makes use of a microcontroller in which the coupling functions (13.7) can be programmed. The use of a microcontroller to define the coupling functions allows relatively easy experimenting with different network topologies and changes in coupling strength.

13.5 Synchronization Experiments

Two systems

Before we discuss synchronization in a network with all four systems, we first consider the case of two diffusively coupled systems being interconnected with coupling strength $K$.

The two systems are connected by feedback (13.7) with $\Gamma$ defined as:

$$\Gamma = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix}.$$
It turns out that the two electronic Hindmarsh-Rose oscillators practically synchronize when \( K \geq 0.6 \). This experimentally obtained coupling strength is pretty close to the value \( K \geq 0.5 \) that is found in simulations. Figure 13.3 shows the practical synchronization of the two systems for \( K = 0.6 \). In the left pane the \( y \)-states of the two systems are shown as function of time. The synchronization phase portrait is depicted on the right pane. The same is shown in Figure 13.4 in case that the coupling between the two systems is twice that large. One can see that the error between both signals decreases when \( K \) increases, i.e. \( \delta \) becomes smaller.

Fig. 13.3 Synchronization of the two coupled systems with \( K = 0.6 \).

Fig. 13.4 Synchronization of the two coupled systems with \( K = 1.2 \).
Three systems

Next, three electronic Hindmarsh-Rose oscillators coupled in a ring are considered. The corresponding coupling matrix is given by:

\[
\Gamma' = \begin{bmatrix}
2K & -K & -K \\
-K & 2K & -K \\
-K & -K & 2K
\end{bmatrix}.
\]

Using a conjecture stated by Wu and Chua (1996), the coupling strength required to synchronize the three systems can be determined from the coupling that is required to synchronize two systems. The conjecture is formulated as follows: given two networks of diffusively coupled systems, if the systems in the network with coupling matrix \( \Gamma \) synchronize, then the systems in the network with coupling matrix \( \Gamma' \) synchronize if and only if \( \gamma = \gamma' \), where \( \gamma \) and \( \gamma' \) denote the smallest nonzero eigenvalues of \( \Gamma \) and \( \Gamma' \), respectively. Applying the Wu-Chua conjecture we expect the systems to synchronize when \( K \geq 0.4 \) in the experimental setup and for \( K \geq 0.34 \) in the simulation. Indeed, the three connected Hindmarsh-Rose oscillators show synchronized behavior for \( K = 0.34 \) in a simulation study, and the three experimental systems in the network practically synchronize when \( K \geq 0.4 \), see Figure 13.5.

**Remark 13.2.** Although it can be shown that the Wu-Chua conjecture is not valid in general, cf. [Pecora, 1998], it is stated in [Pogromsky and Nijmeijer, 2001] that the conjecture is true for systems satisfying assumption A2 of Theorem 13.1.

![Synchronization phase portrait for three systems (K = 0.4).](image)
The four systems are coupled in a ring as shown schematically in Figure 13.6.

The corresponding coupling matrix for this setup is given as

\[
\Gamma'' = \begin{bmatrix}
K_1 + K_2 & -K_1 & 0 & -K_2 \\
-K_1 & K_1 + K_2 & -K_2 & 0 \\
0 & -K_2 & K_1 + K_2 & -K_1 \\
-K_2 & 0 & -K_1 & K_1 + K_2 \\
\end{bmatrix}
\]

The network described by the matrix \(\Gamma''\) does possess some symmetries. The following matrices define a permutation of the network:

\[
\Pi_1 = \begin{bmatrix}
E & O \\
O & E \\
\end{bmatrix}, \quad \Pi_2 = \begin{bmatrix}
O & E \\
E & O \\
\end{bmatrix}, \quad \Pi_3 = \begin{bmatrix}
O & I_2 \\
I_2 & O \\
\end{bmatrix},
\]

and \(\Pi_4 = I_4\), where \(O\) denotes the \(2 \times 2\) matrix with all its elements equal to zero and

\[
E = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}.
\]

The action of \(\Pi_1\) is to switch the systems 1 and 2, and simultaneously switching of the systems 3 and 4. It follows immediately from Figure 13.6 that the network is left invariant with respect to its interconnections. The matrices \(\Pi_2\) and \(\Pi_3\) define similar actions, while \(\Pi_4\) leaves everything unchanged.
The matrices $\Pi_1$, $\Pi_2$ and $\Pi_3$ define, respectively, the following linear invariant manifolds:

\[
\begin{align*}
A_1 &= \{ x \in \mathbb{R}^{12} | x_1 = x_2, x_3 = x_4 \}, \\
A_2 &= \{ x \in \mathbb{R}^{12} | x_1 = x_4, x_2 = x_3 \}, \\
A_3 &= \{ x \in \mathbb{R}^{12} | x_1 = x_3, x_2 = x_4 \}.
\end{align*}
\]

Applying Theorem 13.1, we have $\lambda' = 2K_1$ for $\Pi_1$, $\lambda' = 2K_2$ for $\Pi_2$, and $\lambda' = \min(2K_1, 2K_2)$ for $\Pi_3$. This means that for large enough $K_1$ we can expect asymptotic stability of a subset of $A_1$ and for large enough $K_2$ a subset of $A_2$ is asymptotically stable. A subset of $A_3$ can only be stable as the stable intersection of $A_1$ and $A_2$, which describes the fully synchronized state. In our experimental setup practical partial synchronization with respect to the manifold $A_1$ is found for $K_1 \geq 0.6 > K_2$, while practical partial synchronization with respect to $A_2$ follows, obviously, when $K_2 \geq 0.6 > K_1$. The phase portraits corresponding to these synchronization regimes are depicted in Figures 13.7 and 13.8, respectively.

Fig. 13.7 Partial practical synchronization of four systems with respect to the linear invariant manifold $A_1$ ($K_1 = 0.6$, $K_2 = 0.3$).
Fig. 13.8 Partial practical synchronization of four systems with respect to the linear invariant manifold $A_1 (K_1 = 0.3, K_2 = 0.6)$.

Depending on the values of $K_1$ and $K_2$, in this ring setup there are two possible routes from no synchronization to full synchronization:

- $\text{no synchrony} \rightarrow A_1 \rightarrow A_1 \cap A_2$ (full synchrony),
- $\text{no synchrony} \rightarrow A_2 \rightarrow A_1 \cap A_2$ (full synchrony).

The systems in the experimental setup will indeed practically synchronize when $K_1 = K_2 \geq 0.6$. Figure 13.9 shows the synchronization phase portraits for the four systems in case that $K = 0.6$.

13.6 Conclusions

We have presented our experimental finding of synchronous behavior in an experimental setup with up to four coupled electronic, chaotic Hindmarsh-Rose oscillators. At first, it is shown that each free Hindmarsh-Rose system is semipassive and the internal dynamics are convergent. Therefore, under the condition that the coupling between the systems is large enough, the
systems in the network should show (partially) synchronized behavior. Indeed, in the experimental setup synchronization (and partial synchronization) in networks consisting of two, three or four oscillators is witnessed. We remark that because of small differences in the behavior of the individual circuits, we are not able to achieve a perfect zero synchronization error, but practical (partial) synchronization is achieved.

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