Simulating wave propagation through band gap structures

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Traineeship report

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Abstract

Structures can be exposed to waves for example, light waves, acoustic waves and vibrations. In some cases it can be harmful when these waves are transmitted through the structure. There are some methods to reduce these waves. For example, by adding a band gap structure or varying the material properties. These reduction methods are not enough to attenuate light waves, acoustic waves and vibrations. The idea is to combine the above two reduction methods, so an elastic rod with a band gap structure which moves with a velocity will arise. The purpose of this research is to analyze and try to manipulate the wave propagation through an elastic rod. An elastic rod with a band gap structure which moves with a velocity $V$ will be studied. Furthermore an elastic rod with one inclusion of another material which moves with a velocity $V_1$ at the front and a velocity $V_2$ at the rear will be studied.

A finite element model is used to simulate the wave propagation through an elastic rod. The following cases are simulated: an elastic rod with a static band gap structure, an elastic rod with a band gap structure which moves with a velocity and an elastic rod with one inclusion which moves with velocity $V_1$ at the front of the inclusion and a velocity $V_2$ at the rear of the inclusion.

The band gap can be manipulated by changing the parameters of the structure. A velocity $V$ exposed to a band gap structure causes a wavenumber shift of the reflected wave. The center-frequency of the reflected wave can be manipulated by changing the velocity $V$. The last simulation shows that it is also possible to manipulate the center-frequency of the transmitted wave.
Preface

This report is written for an abroad master traineeship of the Eindhoven University of Technology. This traineeship has been done at the Technical University of Denmark. During the traineeship a study is done on the subject of numerical simulation of dynamic structures.

Some people helped me to arrange this abroad master traineeship. Professor Henk Nijmeijer has made the contacts for this traineeship and I am very grateful for that. Also I want to thank professor Jon Juel Thomsen for the opportunity to do my master traineeship at the Technical University of Denmark.

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Chapter 1

Introduction

1.1 General background

A lot of structures are exposed to waves for example light waves, acoustic waves and vibrations that propagate through the structure. Sometimes it is not desirable that the structure transmits these waves and in some cases it is sufficient to attenuate a single frequency or a frequency range. It is possible to reduce or limit the wave propagation through a structure by adding a band gap structure like shown in figure 1.1. A band gap structure is a periodic structure where partitions of a different material (a stiffer or weaker material) are added. Figure 1.1 depicts a static band gap structure, if the velocity $V$ is 0 [m/s]. The inclusions of the structure will reflect the propagating wave, so some frequencies are attenuated. This means that a structure with band gap properties will attenuate or reduce transmission of propagating waves, for example acoustic waves, light waves and vibrations. The performance of the band gap structure depends on the material properties and geometry of the inclusions. Another way to attenuate or in general manipulate the wave propagation is by varying material properties in time. In this report these two effects will be combined.

The purpose of this report is to analyze and manipulate the wave propagation through a band gap structure with a velocity $V$ in [m/s]. A schematic model of this structure is depicted in figure 1.1. Furthermore this research is done to analyze the wave propagation through an elastic rod with one inclusion that moves with a velocity $V_1$ at the front and a velocity $V_2$ at the end of the inclusion.

![Figure 1.1: An elastic rod with a band gap structure (with 8 inclusions) that moves with a velocity $V$ in [m/s].](image)

Figure 1.1: An elastic rod with a band gap structure (with 8 inclusions) that moves with a velocity $V$ in [m/s].
1.2 Literature review

In the past a lot of research has been done on band gap structures. Years ago Kushwaha [8] has given an extensive review of research in band structures of periodic materials. Shen and Cao [16] have done some research on acoustic band gap engineering, especially in acoustic band gaps that have been experimentally observed in one-dimensional structures. Scalora et al. [15] published a paper about how to design a structure to block a specific light wave. Research on band gap materials has been done by López [9]. Active constrained layer damping is studied by Soons [17]. She compared the active constrained layer damping with passive constrained layer damping. Band gap phenomena have a broad range of interesting applications.

Some applications of a band gap structure are:

- elastic wave attenuation;
- acoustic wave attenuation;
- control of electromagnetic wave propagation in optical waveguides.

The last few years a lot of research has been done on topology optimization of band gap structures. The paper of Dahl et al. [4] describes the topology optimization for transient wave propagation problems in one dimension. The structures in this paper will be used for the analysis in this report.

Another way to manipulate wave propagation is to have material properties that vary over time. For example the stiffness can be changed at a specific time or by adding a velocity to the structure. There are a lot of studies with time varying material parameters in the dynamics of structures. An instant change in stiffness can be used for vibration control. A numerical demonstration is presented by Clark [2], who studied how the vibrations are controlled by changing the stiffness of the structure by applying piezoelectric materials. Ramaratnam and Jalili [12] demonstrated vibration control theoretically and experimentally by using a bi-stiffness spring setting. A similar problem was considered by Issa et al. [5] where a controllable hinge switched the stiffness and showed numerical and experimental vibration attenuation. In most cases the stiffness is switched for controlling vibrations. Blekhman and Lurie [1] have studied the general properties of space-time varying materials, also called “dynamic materials”. Another example of dynamic materials is in the work of Lurie [10]. Weekes [19] predicted the amplitudes of the forward traveling wave (transmitted wave) and the backward traveling wave (reflected wave) when the stiffness of the structure is changed instantly. Furthermore Weekes [18] studied wave propagation in structures with a moving material interface. It shows how to calculate the relative amplitudes of the transmitted wave and the reflected wave. In the research of Jensen [7] a wavenumber shift of the reflected waves can be
calculated. This also is called Doppler shift. A reverse Doppler effect is studied by Reed et al. [13]. Also Reed et al. [14] studied reversed and anomalous Doppler effects in photonic crystals and other time-dependent periodic media.

In the last years a lot of studies have been done to optimize the band gap structure, so the propagation waves attenuate better. The work of Jensen [7] is an extension of the topology optimization concept for transient loads of Dahl et al. [4] with design variables in temporal domain.

1.3 Problem statement

It is desirable to design a structure that attenuates the propagating waves better. A reason is to reduce vibrations of machines, especially when the machines make products with very small tolerances (of nanometer size). Also acoustic noise could be a reason to attenuate the propagating waves in a structure. The problem is how to get a better reduction of the waves. No studies have been made about what will happen with the wave propagation when a band gap structure will be exposed to a velocity. In this report these two described effects will be combined. So a band gap structure which moves in time will be analyzed. This means that at a specific time the band gap structure is exposed to a velocity and provides a change of the stiffness matrix. So the research will be an extension of the work done on band gap structures and time varying material properties.

The main purpose of this report is to analyze the wave propagation through an elastic rod with a band gap structure when the band gap structure will impose a speed. This report explains what the effects of the wave propagation are, when a velocity is exposed to the band gap structure. Another purpose of this report is to study the possibility to manipulate the transmitted and reflected waves. For the calculations, normalized values of material properties are used. These properties can easily be changed to the real material properties. To make clear what the principle is, a simple elastic rod model will be used as a mechanical system. Subsequently, this model will be extended to an elastic rod with a band gap structure which is exposed to a velocity. Furthermore in this report a elastic rod with one inclusion will be analyzed. The inclusion will get a velocity $V_1$ at the front and a velocity $V_2$ at the rear of the inclusion. This simulation is done to see what will happen with the wave propagation when two velocities are added to the inclusion.

1.4 Organization of the report

The report is composed as followed. In chapter 2, the simulation method for a simple elastic rod will be described. In the first section of this chapter the equations for a one-dimensional wave propagation will be defined and the boundary
equations will be presented. Subsequently, the physical rod model will be transformed to a mathematical model. An explanation will be given how to calculate the mass matrix, damping matrix and stiffness matrix. In the next section the input load will be defined. Subsequently, the simulation procedure is given to solve the mathematical model. In chapter 3 the results of three different cases will be presented. The first case is a simulation of an elastic rod with a static band gap structure. Subsequently, the results of an elastic rod with a band gap structure which moves with a velocity $V$ will be presented. The last section presents the results of an elastic rod model with one inclusion which moves with a velocity $V_1$ at the front of the inclusion and a velocity $V_2$ at the rear of the inclusion. All these simulations will be explained what the influences are when changing some parameters. Finally the conclusions and recommendations are presented.
Chapter 2

Simulation of a rod model

This chapter will clarify the principle of the simulation method for the simple elastic rod model. In the first section of this chapter a physical rod model is described. Furthermore the two boundary conditions are presented. Subsequently, the physical rod model will be transformed to a mathematical model. In this section the different matrices (mass matrix, damping matrix and the stiffness matrix) will be presented. The input load for the simulation will be defined in the next section. After these steps the mathematical model is ready for simulation. Subsequently, the solution method will be presented. At the end of this chapter some adjustments will be made so the final model can be simulated.

2.1 A physical rod model

The method for simulating wave propagation through an elastic rod with a band gap structure, which moves with a velocity $V$, is the same as for an simple elastic rod. For the simplicity the method for a simple elastic rod model will be described below, so a physical rod model will be used. The developed model can also describe acoustic and optical wave propagation. In this chapter, longitudinal wave propagation in an infinite linearly elastic rod will be considered.

Figure 2.1: Schematic representation of the physical problem with absorbing boundary conditions at both ends.
The wave equation for heterogeneous media can be obtained by using Newton’s second law \( F = m \cdot a \) with \( a \) the acceleration) and Hooke’s law \( \sigma = E(x)\epsilon \). In appendix A the derivation of the wave equation is shown

\[
\rho(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left( E(x) \frac{\partial u}{\partial x} \right)
\]  

(2.1)

In this equation, \( \rho(x) \) is the mass density in \([\text{kg/m}^3]\), \( E(x) \) is the Young’s modulus in \([\text{N/m}^2]\) and \( u(x, t) \) is the displacement at point \( x \) in \([\text{m}]\). The equation is a linear partial differential equation, because \( u(x, t) \) and its derivatives only occur in linear combinations. In this report a time-limited pulse response is considered. This means that the physical problem will be limited to the time interval \( t \in [0; T] \).

To transform the physical rod model (2.1) to a finite element model, two boundary conditions are required. Absorbing boundary conditions are added to the elastic rod model in order to simulate wave propagation through the elastic rod. So a viscous damper will be added on each end of the elastic rod. An infinite waveguide can be modeled when the absorbing boundary conditions are applied to a finite waveguide by impedance matching. In figure 2.1 the physical problem with the boundary conditions is depicted.

The boundary condition at the left end of the finite element model is

\[-\frac{\partial u}{\partial x} + \frac{1}{c} \frac{\partial u}{\partial t} = \frac{f(t)}{EAc}\]

(2.2)

The boundary condition at the right end of the finite element model is

\[\frac{\partial u}{\partial x} + \frac{1}{c} \frac{\partial u}{\partial t} = 0\]

(2.3)

In (2.2) and (2.3), \( A \) is the local cross-section area of the rod in \([\text{m}^2]\), \( c \equiv \sqrt{\frac{E}{\rho}} \) is the local speed in the material in \([\text{m/s}]\) and \( f(t) \) is the time-varying input load in \([\text{N}]\). The derivation of the boundary conditions are presented in appendix B. The absorbing terms in the boundary conditions (2.2) and (2.3) correspond to optimal viscous dampers. The value of the damping coefficients of these viscous dampers are calculated in appendix B. The time-varying input load will be added at the left end of the finite element model (see figure 2.1). Later on in this report the input load will be defined.

### 2.2 Mathematical model

The physical rod model needs to be transformed into a mathematical simulation model. In Cook et al. [3] a general derivation of a dynamical structure is
given. In the same way the model given in (2.1) can be rewritten to a standard finite element model. By discretization the physical model can be transformed into a mathematical model. The discretization ensures that the computational domain will be divided into \( N_{el} = N - 1 \) linear elements (with \( N \) the number of nodes), so a standard finite element model arises. This standard finite element model can easily be solved. How to solve the mathematical problem will be described in this chapter later. The obtained mathematical model is a system of second-order ordinary differential equations

\[
M \ddot{u} + C \dot{u} + Ku = F(t), \quad t \in [0; T]
\]

In (2.4) \( M, C \) and \( K \) are the usual \( N \times N \) mass, damping and stiffness matrices, respectively. The nodal displacement vector is \( u \) in \([m]\) and the vector \( F(t) \) is the time dependent load in \([N]\).

Equation (2.4), is similar to a standard dynamical structure of finite elements. This means that the mass matrix and sometimes the damping matrix (which depends on the damping model) have the same sparse topology as the stiffness matrix. Another remark regarding this system is that the mass, damping and stiffness matrices can be computed element-wise. So it is easy to calculate these matrices. The only question is how to assemble the mass, damping and stiffness matrices. The upcoming subsections explain how to construct the element matrices.

### 2.2.1 Mass matrix

The mathematical model is obtained by discretization. This means that in (2.4) the mass matrix is a discrete representation of a continuous mass distribution. The mass matrix \( M \) can be assembled element-wise. There are three different types of mass matrices:

- lumped mass matrix;
- consistent mass matrix;
- combination mass matrix.

Each method to compute the mass matrix has its own advantages and disadvantages. A property of a lumped mass matrix is that the element mass matrix is diagonal. This results in less storage space and processing time. Also the inverse of the mass matrix \( M \) is easy to compute and inexpensive. A drawback of a lumped mass matrix is that in some cases the solution is less accurate. When looking at the consistent mass matrix and the combination mass matrix the element mass matrix is no longer diagonal, so the calculation of the inverse is time consuming and expensive.
For the simulation of the problem it is important that the calculations are fast. So in this report a particle mass lumping method will be used to calculate the mass matrix of the mathematical problem. The other element mass matrices are given in appendix C. In figure 2.2 an element of the elastic rod is depicted. The total mass of the element is \( m = \rho AL \), where \( \rho \) is the mass density in \( [kg/m^3] \), \( A \) is the cross-sectional area of the rod in \( [m^2] \) and \( L \) is the element length in \( [m] \). By particle lumping the total mass will be divided over the two nodes, so each node will have a mass of \( m/2 \). Figure 2.2 shows a two-node elastic rod with the axial displacements and the divided mass. The corresponding element mass matrix is:

\[
[m] = \frac{m}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2} \rho AL \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]  

(2.5)

The matrix (2.5) is an element mass matrix for one-dimensional problems. The one-dimensional problem gives the direction of the displacements. In this problem, the displacements are in axial direction (see figure 2.2). By using matrix (2.5) the global mass matrix can be computed. When the mass density \( \rho \) of the whole elastic rod is the same, the \((N \times N)\) global mass matrix becomes

\[
M = \frac{1}{2} \rho AL
\begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & 2 & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \\
\vdots & \ddots & \ddots & 2 & 0 \\
0 & \cdots & \cdots & 0 & 1
\end{bmatrix}
\]  

(2.6)

2.2.2 Damping matrix

The damping matrix \( C \) is very easy to compute. Only at each end a viscous damper is applied to the elastic rod. This means that on the first entry (first node) and last entry (last node) of the finite element model, the damping coefficient will be placed in the global damping matrix. The damping coefficients of the two viscous dampers are equal to \( A\sqrt{E\rho} \) in \([Ns/m]\), where \( A \) is the cross-sectional area in \([m^2]\), \( E \) is the Young’s modulus in \([N/m^2]\) and \( \rho \) is the mass density in \([kg/m^3]\) \((E, A \text{ and } \rho \text{ are the material properties of the first and last element respectively})\). For the derivation of the damping coefficient
see appendix B. The \((N \times N)\) global damping matrix becomes
\[
C = A\sqrt{E \rho} \begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 0 & 0 & \vdots \\
0 & \cdots & \cdots & 0 & 1
\end{bmatrix}
\]
(2.7)

### 2.2.3 Stiffness matrix

Cook et al. [3] and Peerlings [11] present a method to compute the element stiffness matrix and assemble the stiffness matrix \(K\). For computing the element stiffness matrix a two-node rod element will be considered which only allows a displacement in axial direction. By using a free-body diagram (see figure 2.3), the axial stress \(\sigma\) can be related to the nodal forces \(F_1\) and \(F_2\). Also the axial stress \(\sigma\) is related to the Young’s modulus \(E\) and the dimensionless axial strain \(\epsilon = \Delta L/L = (u_2 - u_1)/L\).

![Two-node rod element](image)

Figure 2.3: Two-node rod element with a cross-sectional area \(A\) in \([m^2]\) showing internal stress \(\sigma\) in \([N/m^2]\).

The calculation of the equilibrium equations is done using the free-body diagram (see figure 2.3). In [3] the derivations of the equilibrium equations are presented (see appendix D). Using these equations the element stiffness matrix becomes
\[
[k] = \begin{bmatrix}
k & -k \\
-k & k
\end{bmatrix}
\text{ with } k = \frac{AE}{L}
\]
(2.8)

The coefficient \(k\) can be regarded as the stiffness of a linear spring. So an elastic rod has the same behavior as a linear spring. The factor \(k\) depends on the material properties of the element. This means that the element stiffness matrix \([k]\) may change for each element. When the element stiffness matrix \([k]\) is known for each element, the stiffness matrix can be calculated. The stiffness matrix has the same size as the mass matrix \(M\) and the damping matrix \(C\). If the Young’s modulus varies over time, the stiffness matrix \(K\) becomes time
dependent, yielding $K(t)$. For the simulation it is not necessary to compute the whole stiffness matrix $K$, because most of the time there is only interest in the displacement of a particular node. This makes the calculation easier since it is sufficient to use the positions in the matrix that correspond with the particular node. These positions are easy to compute with the element stiffness matrix.

### 2.3 Input load

In this section the input load will be defined. In (2.2) an input load $f(t)$ is specified on node 1. By using Gaussian wave packets a well-defined excitation in the frequency domain will be obtained. This input signal results in a “nice” bell curve in the frequency domain, with a center-frequency of $\omega$ in [rad/s] (see figure 2.9 the smooth curve in the frequency domain). The attenuation of the input signal is clearly visible when the band gap structure is placed in the correct frequency range. The equation of the Gaussian wave packets is

$$f(t) = 2\mu_0\omega_0 \cos(\omega_0(t-t_0))e^{-\delta(t-t_0)^2} \tag{2.9}$$

In this equation $\mu_0$ is the amplitude of the generated wave pulse in [m], $\omega_0$ is the center-frequency of the excitation force in [rad/s], $t$ is the time in [s], $t_0$ is the time where the center of the wave packet is in [s] and $\delta$ is the width of the

![Gaussian wave packet excitation](image)

Figure 2.4: Gaussian wave packet excitation with the parameters $t_0 = 2.5$ [s], $\omega_0 = 18.5$ [rad/s], $\mu_0 = 1$ [m] and $\delta = 1$ [s$^{-2}$], in the time domain (top) and in the frequency domain (bottom).
wave pulse in \([s^{-2}]\). In figure 2.4 a Gaussian wave pulse is depicted in the time
domain and in the frequency domain.

For the mathematical model (2.4) matrix \(F(t)\) is needed. All the nodal forces
are placed in the matrix \(F(t)\), so (2.9) is inserted at the first entry of this matrix.
The form of matrix \(F(t)\) is

\[
F(t) = \begin{bmatrix}
f(t) \\
0 \\
\vdots \\
0
\end{bmatrix}
\]  

(2.10)

where \(f(t)\) is defined in (2.9).

2.4 Solution method

By integration solutions of (2.4) can be obtained. Cook et al. [3] worked out
different methods to solve this kind of problems. One way to integrate the
mathematical model is by using direct integration. When using direct integra-
tion the response history will be calculated step-by-step in time. There is no
need to change the form of the dynamic equations, as is necessary when us-
ing modal methods. The response is evaluated at each time step \(\Delta t\). So the
equation of motion (2.4) for the \(n\)th time step is

\[
M\ddot{u}_n + C\dot{u}_n + Ku_n = F(t)_n
\]  

(2.11)

By using finite difference approximations of time derivatives a discretization in
time can be attained. The algorithms for the direct integration can be classified
as explicit or implicit. When using the explicit algorithm only past information
is needed, but when using the implicit algorithm future information is needed.
In this report the explicit direct integration is chosen because it is best suited for
wave propagation problems and easy to implement. Advantages of the explicit
direct integration are low computer storage requirements and low costs per
time step. A drawback is that explicit methods are only conditionally stable.
This means that there is a “critical” time step \(\Delta t_{cr}\) that must not be exceeded,
otherwise the algorithm is unstable and no solution will be obtained.

2.4.1 Explicit direct integration

To estimate the time derivatives for each step \(\Delta t\) a finite difference approxi-
mation will be used (see Cook et al. [3]). There are separate forms of the central dif-
fERENCE method, for example the classical central differences and the half-step
central differences. For this problem the classical central differences method
is used (for the derivation see appendix E), for solving (2.4). The displacement
vector at the next time step \(u_{n+1}\) is based on the displacement vectors \(u_{n-1}\) at
the previous time step and \(u_n\) at the current time step.
The displacement vector \( u_{n+1} \) is approximated as
\[
\frac{1}{(\Delta t)^2} Mu_{n+1} \approx F_n - Ku_n + \left( \frac{2}{(\Delta t)^2} M - \frac{1}{\Delta t} C \right) u_n - \left( \frac{1}{(\Delta t)^2} M - \frac{1}{\Delta t} C \right) u_{n-1}
\] (2.12)
where \( F_n \) is the load vector at the current time step. The mass matrix (see (2.6)) is diagonal, so the inverse is easy to compute. This means that it is easy to compute the displacement vector \( u_{n+1} \).

For solving (2.12) for the first time step \( n = 0 \) a problem occurs. In this equation \( u_{-1} \) is unknown. The unknown \( u_{-1} \) can be calculated with
\[
u_{-1} = u_0 - \Delta t \dot{u}_0 + \frac{(\Delta t)^2}{2} \ddot{u}_0
\] (2.13)

By evaluating (2.11) at \( n = 0 \) the second time derivative \( \ddot{u}_0 \) can be obtained by
\[
\ddot{u}_0 = M^{-1} (F_0 - Ku_0 - Cu_0)
\] (2.14)
In this equation \( F_0, u_0 \) and \( \dot{u}_0 \) are the initial values. Because the elastic rod is chosen to be in rest at \( t = 0 \) [s] the initial values of (2.14) are \( u_0 = 0, \dot{u}_0 = 0 \) and \( F_0 = 0 \).

### 2.4.2 Stability

As mentioned before an explicit method is conditionally stable, which means that the displacements will grow to infinity if the step size \( \Delta t \) is too large. A too large time step results in an unbounded solution that may grow by orders of magnitude per time step and due to overflow the simulation will stop. When \( \Delta t \) is smaller or equal to a “critical” time step \( \Delta t_{cr} \) (2.12) is stable. The \( \Delta t_{cr} \) is related to the maximal natural frequency \( \omega_{max} \) and its period \( T_{min} \) that is shown in (2.15):
\[
\Delta t \leq \frac{2}{\omega_{max}^2} = \Delta t_{cr} \quad \omega = 2\pi f = \frac{2\pi}{T} \quad \text{so} \quad \Delta t \leq \frac{T_{min}}{\pi}
\] (2.15)

The maximal natural frequency and the minimal period time can be calculated from \( (K - \omega^2 M)u = 0 \). The highest natural frequency corresponds to \( \omega_{max} \). This method is only used when the finite element model contains a few elements, otherwise it is time consuming and expensive. There are other ways to estimate \( \omega_{max} \). For the elastic rod model, the highest frequency for lumped masses is
\[
\omega_{max} = 2 \sqrt{\frac{AE}{mL}} \quad \text{with} \quad m = \rho AL \quad \text{yields} \quad \omega_{max} = 2 \sqrt{\frac{E}{L}} \frac{2\pi}{L} = \frac{2c}{L}
\] (2.16)
In this equation $c = \sqrt{E/\rho}$ is the speed of sound in the material in \([m/s]\). A different method to calculate $\omega_{\text{max}}$ is by using the Gershgorin bound (see [3]). For lumped mass matrices the equation is

$$\omega_{\text{max}}^2 \leq \max_i \left( \frac{1}{M_{ii}} \sum_{j=1}^{n} |K_{ij}| \right) \quad \text{where} \quad i = 1, 2, \ldots, n \quad (2.17)$$

### 2.5 Final model

The final model of this report considers an elastic rod with inhomogeneous material properties that may vary in time. In the previous sections the solution strategy is presented of a simple elastic rod. The same strategy will be used for the final model. Minor adjustments are required to simulate the final model. First it is needed to define the material properties of each element. There is chosen to consider that the mass density of the elastic rod and the inclusions are the same. The only difference between the elastic rod material and the inclusion material is the Young’s modulus $E$. Now the model (2.4) will change to

$$M\ddot{u} + C\dot{u} + K(t)u = F(t), \quad t \in [0; T] \quad (2.18)$$

In this equation the mass matrix $M$ and damping matrix $C$ are constant (with $M$ is matrix (2.6) and $C$ is matrix (2.7) respectively), but the stiffness matrix $K$ will change in time. This means that the stiffness matrix needs to be updated, because the material properties of each element can change. Because the velocity is added to the band gap structure, the solution method (2.11) will change to

$$M\ddot{u}_n + C\dot{u}_n + K_n u_n = F_n(t) \quad (2.19)$$

This equation can be solved via

$$\frac{1}{(\Delta t)^2} M u_{n+1} \approx F_n - K_n u_n + \left( \frac{2}{(\Delta t)^2} M - \frac{1}{\Delta t} C \right) u_n$$

$$- \left( \frac{1}{(\Delta t)^2} M - \frac{1}{\Delta t} C \right) u_{n-1} \quad (2.20)$$

where $K_n$ is the stiffness matrix at the current time step.

It is possible that the results show some spurious oscillations. If these oscillations are too large then it is needed to suppress them. Jensen [7] has described a way to reduce the disturbance behavior, without loss of the overall behavior. By adding stiffness dependent damping the spurious oscillations are dissipated. The damping mainly reduces the high-frequency oscillations. The stiffness dependent damping is calculated as

$$C_{\text{extra}} = \frac{\beta}{\omega_0^2} \tilde{K} \quad (2.21)$$
In this equation $\omega_0$ is the center-frequency of the input load [rad/s], $\beta$ is the damping coefficient in [rad$^2$/s] and $\tilde{K}$ is a constant stiffness matrix of the elastic rod material (without inclusions) in [N/m]. The new $(N \times N)$ global damping matrix is

$$C = C_{viscous} + C_{extra}$$

(2.22)

where $C_{viscous}$ is defined in (2.7).
Chapter 3

Simulation results

The previous chapter shows the solution strategy of the main problem of this report. In this chapter the simulation results of three different cases will be presented. The three different cases are:

- an elastic rod with a static band gap structure;
- an elastic rod with a band gap structure which moves with a certain velocity $V$;
- an elastic rod with one inclusion which moves with a velocity $V_1$ at the front of the inclusion and a $V_2$ at the back of the inclusion.

A schematic representation of the different cases is depicted in figure 3.1. Figure 3.1 shows where the velocity $V$ is added to the band gap structure. The solution method of chapter 2 is implemented in Matlab. Subsequently, the three different cases can be simulated. In the next sections the three different cases with standard settings are simulated.

![Figure 3.1: The different simulation cases (a) band gap structure, (b) band gap structure with $V$ and (c) one inclusion with $V_1$ and $V_2$.](image)
3.1 A static band gap structure

In this section an elastic rod with a static band gap structure with standard settings is simulated. When using an elastic rod with a static band gap structure, the stiffness matrix \( K \) is constant. The other matrices (\( M \) and \( C \)) are calculated as before. For the simulation, the well-defined input load of section 2.3 will be used (see figure 2.4), with the parameters \( t_0 = 2.5 \ [s] \), \( \omega_0 = 18.5 \ [rad/s] \), \( \mu_0 = 1 \ [m] \) and \( \delta = 1 \ [s^{-2}] \). To separate the reflected wave from the input wave a length of 10 \([m]\) is used for the elastic rod. It contains 14 inclusions with a width of 0.1 \([m]\) and between two inclusions a width of 0.1 \([m]\). The elastic rod has a mass density \( \rho = 1 \ [kg/m^3] \) and a Young’s modulus of \( E = 1 \ [N/m] \). The inclusions have the same mass density as the elastic rod and a Young’s modulus of \( E = 2 \ [N/m] \).

In this section a parameter study is carried out. In the first subsection the Young’s modulus \( E \) of the inclusions will be varied. Subsequently, the number of inclusions will be increased and decreased. In the third subsection the inclusion width is changed and in the last subsection the distance between the inclusions will be varied.

3.1.1 Young’s modulus of the inclusions

By using the standard settings above, the frequencies of the input wave are within the band gap. The band gap ensures that some frequencies are attenuated. The other frequencies of the input wave propagate through the elastic rod. A property of the band gap structure is that the frequencies of the input wave always are partly reflected. Small variations in Young’s modulus are used to make clear what happens with the wave propagation when the Young’s modulus changes. If the variation of the Young’s modulus is too large, the center-frequency of input signal does not lie within the band gap created by the band gap structure. The input signal transmits through the band gap structure.

The Young’s modulus of the inclusions is varied between \( E = 1.5 \ [N/m^2] \) to \( E = 2.5 \ [N/m^2] \). Figure 3.2 shows the different graphs of the reflected frequencies, transmitted frequencies and the maximum amplitude of the reflected and the transmitted wave (measured in the time domain). These graphs clearly show what will happen with the propagating wave when increasing the Young’s modulus of the inclusions. The reflected and transmitted waves are obtained by using fast Fourier transformation, which transforms the propagating wave in the time domain to the frequency domain. On the vertical axis of the top and the middle graphs the power is given. This amplitude corresponds to the absolute value of the fast Fourier transformation. Note that the shapes of these figures are important. The first graph shows that when the Young’s modulus increases the amplitude of the reflected wave increases. When looking at figure 3.2, it shows that for \( E = 1.5 \), \( E = 2 \) and \( E = 2.5 \ [N/m^2] \) the
center-frequency of the band gap will be at 17.5, 18 and 20 \([\text{rad/s}]\) respectively. The band gap shifts to the left when \(E\) is decreased and shifts a little bit to the right when \(E\) is increased (compared to the standard \(E = 2 \text{ [N/m}^2\text{]}\)).

![Reflected frequencies](image1)

![Transmitted frequencies](image2)

![Amplitude reflected (R) and transmitted (T) wave](image3)

Figure 3.2: Reflected frequencies (top), transmitted frequencies (middle) and the maximum amplitude of the reflected and transmitted wave (bottom).
The last graph of figure 3.2 clearly shows the amplitude change of the reflected and transmitted wave when \( E \) is changed. For a small Young's modulus the amplitude of the reflected wave is small. When increasing the stiffness of the inclusions the band gap structure will become more rigid. This is why the amplitude of the reflected wave will increase, simultaneously the amplitude of the transmitted wave will decrease (see figure 3.2).

### 3.1.2 Number of inclusions

In this part the number of inclusions of the band gap structure will be investigated. The band gap structure of the standard simulation model has 14 inclusions. The simulation is done for 5 and 25 inclusions, to study the influence of the band gap structure on the input wave. In figure 3.3 the results of the 3 simulations are depicted.

This figure shows what will happen with the input wave that propagates through the elastic rod when the number of inclusions is increased. The second graph of figure 3.3 shows the decrease of the transmitted amplitude of the wave in the frequency domain. The graph clearly shows when the number of inclusions in the band gap structure is increased that the valley of the band gap will be wider and deeper. In the last graph, the amplitude of the reflected wave will be higher when the number of inclusions will increase, but the amplitude of the transmitted wave will decrease. At 13 inclusions the maximum amplitude of the transmitted wave will decrease very slowly. The opposite will happen with the amplitude for the reflected wave. The amplitude of the reflected wave will increase very fast in the beginning, but at 13 inclusions the amplitude of the reflected wave will increase very slowly. An explanation for this phenomena is, that the wave will be reflected.

In the last graph, the amplitude of the reflected wave will be higher when the number of inclusions increases, while the amplitude of the transmitted wave decreases. At 13 inclusions the maximum amplitude of the transmitted wave will decrease very slowly. The opposite will happen with the amplitude for the reflected wave. This wave will increase very fast for few inclusions, but from 8 inclusions, the increase of this amplitude declines. An explanation for this phenomenon is, that every time a wave encounters an inclusion, it will partially reflect and transmit the wave. At a specific number of inclusions the input wave is almost fully reduced and a further increase results in a very small transmitted amplitude.
Figure 3.3: Reflected frequencies (top), transmitted frequencies (middle) and the maximum amplitude of the reflected and transmitted wave (bottom).

3.1.3 Inclusion width

Another parameter that can be changed is the inclusion width. In the standard simulation the inclusion width is set to 0.1 [m]. Figure 3.4 shows the influence of the inclusions width.
Figure 3.4: Reflected frequencies (top), transmitted frequencies (middle) and the maximum amplitude of the reflected and transmitted wave (bottom).

This figure shows that a small change in inclusion width has a huge influence on the center-frequency of the band gap. A lower inclusion width than the standard inclusion width leads to a higher center-frequency of the band gap. This means that higher frequencies are filtered. If the inclusion width is increased, the band gap frequency will be lower. The last graph of figure 3.4 shows the maximum amplitude of the reflected and transmitted waves in the time domain. The maximum amplitude of the reflected wave will decrease when the inclusion width is larger or smaller than the standard inclusion width. The transmitted wave is increased when changing the standard inclusion width.
to another value. Note that it is coincidence that the maximum amplitude of the reflected wave and the transmitted wave is at exactly 0.1 [m]. When using another standard setting it will change.

### 3.1.4 Distance between two inclusions

In the previous subsection the inclusion width has been varied. In this subsection the distance between two inclusions (gap width) is changed. The standard settings of the gap width is 0.1 [m]. The graphs for three different values of the gap width are depicted in figure 3.5.

**Figure 3.5:** Reflected frequencies (top), transmitted frequencies (middle) and the maximum amplitude of the reflected and transmitted wave (bottom).
The first two graphs are quite similar to the first two graphs in figure 3.4. Also the center-frequency of the band gap will decrease for higher values of the gap width. For a gap width of 0.09 [m] the center-frequency of the band gap is approximately 20 [rad/s] while this frequency is approximately 17 [rad/s] with a gap width of 0.11 [m]. The bottom graph looks also the same as figure 3.4. The amplitude will decrease if the value of the gap width is changed to another value than the standard gap width of 0.1 [m].

3.2 A band gap structure that moves with a velocity \( V \)

In the previous section an elastic rod with a static band gap structure is simulated and various parameters are varied. In this section the band gap structure will move with a constant velocity \( V \) in [m/s]. Every inclusion of the band gap structure has the same velocity. The work of Jensen [7] explains what will happen with the wave propagation when an interface between two materials with a different Young’s modulus moves with a velocity \( V \). This model is shown in figure 3.6. Due to the interface speed \( V \), the reflected and transmitted waves get a wavenumber (or frequency) shift. The wavenumber shift of the reflected wave is also called Doppler shift. It is possible to calculate the relative wavenumber shift of the reflected waves by

\[
\frac{\Delta \gamma}{\gamma_0}_R = \frac{1 + V}{1 - V}
\]

When looking at (3.1), it clearly shows that the wavenumber shift only depends on the interface speed. In this equation the velocity \( V \) is relative to the wave speed in the material. The wave speed corresponds to \( c = \sqrt{E/\rho} = 1 \) [m/s]. Furthermore there are no other material properties in (3.1), so the wavenumber shift is predicted to be independent on the material properties. The question is, if this equation still holds for a moving band gap structure. The simulation results of an elastic rod with a band gap structure which moves with a velocity will be compared to this equation.

![Figure 3.6: A one-dimensional elastic rod with a moving material interface.](image)

The simulation is done with the standard settings. The solution results can be compared to the elastic rod with a static band gap structure. To get correct results, it is needed to expose the whole input wave to the moving band gap structure. This means that the band gap structure is moving till the input
wave passes the whole band gap structure. The simulation is done for different values of $V$. The number of inclusions and the velocity $V$ are limited by the length of the elastic rod model. For this simulation the velocity $V$ is varied from $V = 0$ to $V = 0.3\, \text{m/s}$. For higher velocity the first inclusion would reach the beginning of the elastic rod. This is not desired in this study, so this behavior is avoided.

Figure 3.7: Reflected frequencies (top), transmitted frequencies (middle) and the maximum amplitude of the reflected wave (bottom).
The graphs for $V = 0.05$, $V = 0.1$, $V = 0.2$ and $V = 0.3 \text{ [m/s]}$ of the reflected and transmitted frequencies are depicted in figure 3.7. The maximum amplitudes of the reflected waves are given for each velocity and number of inclusions.

When the first two graphs are examined, the wavenumber shift of the reflected and transmitted waves is clearly visible. When the velocity is increased, the wavenumber of the reflected waves shifts to a higher frequency. This is also predicted by (3.1). The last graph shows that the maximum amplitude of the reflected wave in time domain will decrease when the velocity $V$ increases. At approximately $V = 0.2 \text{ [m/s]}$ the amplitude will not decrease anymore, when the number of inclusions is increased. It is possible that for higher velocities the band gap structure can be seen as one big inclusion and then there is a small reflection amplitude. The number of inclusions has almost no influence on the maximum amplitude of the reflected wave. When increasing the number of inclusions between 1 and 4 the amplitude will change. Increasing further has no influence anymore.

In table 3.1 the factor between the frequency of the input signal and the frequency of the reflected wave is shown. For small $V$ the calculated factor by (3.1) is almost the same as the factor of the simulation. At higher velocities the factor varied a little bit to a maximum of 8.36 percent. The reflected signal has more peaks (see figure 3.7) at higher velocities. It is difficult to estimate the center-frequency of the reflected waves. This may cause the differences at higher velocities.

<table>
<thead>
<tr>
<th>Velocity in [m/s]</th>
<th>0.00</th>
<th>0.02</th>
<th>0.04</th>
<th>0.06</th>
<th>0.08</th>
<th>0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3.1)</td>
<td>1</td>
<td>1.0408</td>
<td>1.0833</td>
<td>1.1276</td>
<td>1.1739</td>
<td>1.2222</td>
</tr>
<tr>
<td>simulation</td>
<td>1.0037</td>
<td>1.0422</td>
<td>1.0811</td>
<td>1.1104</td>
<td>1.1691</td>
<td>1.1889</td>
</tr>
<tr>
<td>0.12</td>
<td>1.2727</td>
<td>1.3255</td>
<td>1.3809</td>
<td>1.4390</td>
<td>1.5</td>
<td>1.8571</td>
</tr>
<tr>
<td>0.14</td>
<td>1.2175</td>
<td>1.2372</td>
<td>1.3249</td>
<td>1.3539</td>
<td>1.4519</td>
<td>1.7735</td>
</tr>
</tbody>
</table>

Table 3.1: The factors of the wavenumber shift calculated by (3.1) and by simulation (velocity range 0.00 – 0.30 [m/s]).

The center-frequency of the reflected waves $\omega_r$ is equal to 18.49 [rad/s], when the velocity of the band gap structure is zero. In table 3.1 the factors are given, so the differences between the frequencies can be calculated. In figure 3.8 the differences between the frequency of the reflected waves at a
velocity $V$ and the frequency of a static band gap structure is given. This figure presents the values of table 3.1 in a graph. It shows that for small velocities the calculated and the simulated $\Delta \omega_r$ is approximately the same. Also for higher velocities the difference between the calculated and simulated $\Delta \omega_r$ is small.

![Figure 3.8: Difference in $\omega$ when the velocity increases.](image)

### 3.3 One inclusion that moves with $V_1$ (front) and $V_2$ (rear)

In this section an elastic rod with one inclusion which moves with a velocity $V_1$ at the front and $V_2$ at the rear of the inclusion is considered. For this simulation the same input load is used that is described earlier. The length of the elastic rod is $10 \ [m]$ and the Young’s modulus is $1 \ [N/m^2]$. The number of inclusions are set to 1, the inclusion width is set to $2 \ [m]$ and the Young’s modulus is $2\ [N/m^2]$. For the standard simulation the velocities $V_1$ and $V_2$ are set to $0.1 \ [m/s]$ and $0.3 \ [m/s]$ respectively. Also it is necessary to turn on the damping, otherwise the graphs are not smooth enough. For the simulation the stiffness dependent damping (see (2.21)) is used with a damping coefficient $\beta = 0.005 \ [rad^2/s]$. This damping will reduce the high-frequency oscillations. The value of the damping coefficient $\beta$ is very small, so the amplitudes are almost the same.
In the previous section the center-frequency of the reflected waves was shifted. For this simulation the wavenumber shift of the reflected waves can be calculated with (3.1). The purpose of this simulation is to see if it is possible to manipulate the center-frequency of the transmitted waves. First the model with normal parameter settings will be treated when the velocity $V_1$ (front speed of the inclusion) will be increased and $V_2$ is fixed to 0.3 [m/s]. Subsequently, only the velocity $V_2$ (the rear of the inclusion) will be increased and $V_1$ is fixed to 0.1 [m/s]. In the next part the influence of the Young’s modulus will be studied, so an optimal wavenumber shift of the transmitted waves arises. In the last subsection, the inclusion width will be increased so the difference $\Delta V = V_2 - V_1$ can be made larger.

3.3.1 Influence of $V_1$ and $V_2$

In this section the velocities $V_1$ and $V_2$ are studied. In the first simulation the velocity $V_1$ is increased from 0.1 [m/s] to 0.3 [m/s]. Also the maximum amplitudes of the reflected and transmitted waves will be studied. Subsequently, the same will be done with velocity $V_2$.

The results of the simulation are depicted in figure 3.9. The figure shows what will happen with the wave propagation when increasing $V_1$ or increasing $V_2$. When looking at the left figures, the reflected frequencies shift when the velocity $V_1$ increases. For $V_1 = V_2 = V = 0.3$ [m/s] (3.1) holds.

\[
\left( \frac{\Delta \gamma}{\gamma_0} \right)_R = \frac{1 + V}{1 - V} \\
\left( \frac{\Delta \gamma}{\gamma_0} \right)_T = \frac{1 + 0.3}{1 - 0.3} = 1.857
\]  

(3.2)

The frequency shift can be calculated by $\omega_{\text{shift}} = \left( \frac{1 + V}{1 - V} \right) \omega_{\text{shift}}$. The input signal has a center-frequency of $\omega_0 = 18.5$ [rad/s] and a velocity of $V = 0.3$ [m/s]. This yields a frequency shift of $\omega_{\text{shift}} = 34.357$ [rad/s]. The dash-dotted line in the figure has a peak at approximately the same frequency. The transmitted waves have a small frequency shift too. When increasing the velocity $V_1$, the transmitted wave is shifted to the center-frequency of the input wave and this is not desired. In the bottom figure the maximum amplitudes of the reflected waves (R) and the transmitted waves (T) are depicted. The amplitude of the reflected wave is approximately 0.2 [m] and the amplitude of the transmitted wave is approximately 1 [m] and is independent on the velocity $V_1$ for this range.
Figure 3.9: Reflected frequencies (top), transmitted frequencies (middle) and the maximum amplitude of the reflected and transmitted wave (bottom). The figures left: fixed $V_2 = 0.3\ [m/s]$ and the figures right: fixed $V_1 = 0.1\ [m/s]$.

The right figures shows what will happen with the input wave when the velocity $V_2$ is increased from $V_2 = 0.1\ [m/s]$ to $V_2 = 0.3\ [m/s]$. If $V_1 = V_2$ the wavenumber shift of the center-frequency of the reflected wave can be calculated with equation 3.1. The center-frequency of the transmitted wave is shifted away from $\omega_0$. So increasing $\Delta V$ gives a higher frequency shift of the transmitted wave. The transmitted wave has an amplitude of approximately 1 $[m]$ and the reflected wave has an amplitude of 0.2 $[m]$. So the amplitudes of the reflected and transmitted waves are independent of the velocity $V_2$ for this range.
3.3.2 Influence of the Young’s modulus $E$

In this part, the Young’s modulus will be changed. For the simulation the velocities are still set to $V_1 = 0.1 \text{ [m/s]}$ and $V_2 = 0.3 \text{ [m/s]}$ and will not change. Subsequently the Young’s modulus is set to $E = 2$, $E = 0.5$ and $E = 0.25 \text{ [N/m^2]}$. The results of the simulation for the different Young’s modulus are depicted in figure 3.10.

![Reflected frequencies](image1)

![Transmitted frequencies](image2)

Figure 3.10: Reflected frequencies (top) and transmitted frequencies (bottom) with $V_1 = 0.1 \text{ [m/s]}$ and $V_2 = 0.3 \text{ [m/s]}$.

The top figure shows that the primary reflection only depends on the velocity $V_1$. The wavenumber shift of the primary reflected waves can be calculated by (3.1) (with $V = V_1$). When the Young’s modulus of the inclusion is decreased the amplitude of the reflected wave will decrease until $E = 1 \text{ [N/m^2]}$. For a Young’s modulus smaller than $1 \text{ [N/m^2]}$ the amplitude of the reflected wave increases again.

The main purpose of this simulation is to manipulate the transmitted wave. It is possible to get a small frequency shift of the transmitted wave. The velocity
$V_2$ at the end of the inclusion should ensure that the amplitude of the transmitted wave will be the same as the input signal. The graphs in figure 3.11 shows the maximum amplitude of the reflected wave (top) and the transmitted wave (bottom).

![Amplitude reflected wave](image)

![Amplitude transmitted wave](image)

Figure 3.11: The maximum amplitude of the reflected wave (top) and the transmitted wave (bottom) with $V_2 = 0.3 \text{ [m/s]}$.

The figure shows that the amplitude of the transmitted wave is approximately $1 \text{ [m]}$. For the reflected wave there is a difference. For a Young’s modulus of $E = 2 \text{ [N/m}^2\text{]}$ and $E = 0.5 \text{ [N/m}^2\text{]}$ the amplitude is approximately $0.2 \text{ [m]}$, but for $E = 0.25 \text{ [N/m}^2\text{]}$ the amplitude is $0.3 \text{ [m]}$. It seems that an increasing or decreasing Young’s modulus affects the amplitude. When the Young’s modulus of the inclusion is increased to $E = 5 \text{ [N/m}^2\text{]}$, the inclusion becomes stiffer which increases the amplitude of the reflected wave. The amplitude of the reflected wave depends on the ratio between the Young’s modulus of the inclusions and the Young’s modulus of the elastic rod. Note that for
an elastic rod with one inclusion which moves with a velocity $V_1$ (front) and $V_2$ (rear) the energy is not needed to be conserved.

### 3.3.3 Changing the velocity difference $\Delta V$

In the previous subsection it was clear that the smallest Young’s modulus and a large velocity difference $\Delta V$ results in a relatively large shift of the center-frequency of the transmitted wave. An increasing velocity $V_2$ results in a clear wavenumber shift. With the standard settings the velocity $V_2$ can not be increased. When the inclusion width is increased to $s = 5 \text{ [m]}$, the velocity $V_2$ can be set to $0.4 \text{ [m/s]}$. In figure 3.12 the results are shown when the velocity $V_2$ is increased to $0.4 \text{ [m/s]}$.

![Figure 3.12: Reflected frequencies (top), transmitted frequencies (middle) and the maximum amplitude of the reflected and transmitted wave (bottom).](image)

The center-frequency of the primary reflection is almost the same. Looking at the center-frequency of the transmitted wave, it is moving away from the
input center-frequency $\omega_0$, while the maximum amplitude of the transmitted wave (in the time domain) is approximately $1 \text{ m}$. The maximum amplitude of the reflected wave (in the time domain) is almost the same for every velocity $V_2$.

### 3.4 Summary

An elastic rod with a static band gap structure is simulated. Furthermore a parameter study is carried out for this case. Four different parameters are studied and the results of this study are depicted in table 3.4.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Change</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's Modulus $E$</td>
<td>Increase</td>
<td>Higher center-frequency of the band gap, a higher reflected wave amplitude and a lower transmitted wave amplitude.</td>
</tr>
<tr>
<td>Number of inclusions</td>
<td>Increase</td>
<td>Deeper and wider band gap, a higher reflected wave amplitude and a lower transmitted wave.</td>
</tr>
<tr>
<td>Inclusion width</td>
<td>Increase</td>
<td>Lower center-frequency of the band gap, a lower reflected wave amplitude and a higher transmitted wave amplitude.</td>
</tr>
<tr>
<td>Inclusion width</td>
<td>Decrease</td>
<td>Higher center-frequency of the band gap, a higher reflected wave amplitude and a higher transmitted wave amplitude.</td>
</tr>
<tr>
<td>Gap width</td>
<td>Increase</td>
<td>Lower center-frequency of the band gap, a lower reflected wave amplitude and a higher transmitted wave amplitude.</td>
</tr>
<tr>
<td>Gap width</td>
<td>Decrease</td>
<td>Higher center-frequency of the band gap, a lower reflected wave amplitude and a higher transmitted wave amplitude.</td>
</tr>
</tbody>
</table>

Table 3.2: Results of an elastic rod with a static band gap structure.

The propagating input wave can be manipulated by using an elastic rod with a band gap structure. By choosing the right parameters, it is possible to reduce some frequencies of the input wave that transmits through an elastic rod.
In the case of an elastic rod with a band gap structure which moves with a velocity $V$, the velocity $V$ is varied. Also the wavenumber shift is compared with 3.1. This simulation shows that it is possible to get a wavenumber shift of the reflected and transmitted wave. Furthermore, the simulated wavenumber shift is approximately the same as the calculated wavenumber shift.

In the last simulation an elastic rod with one inclusion is simulated. Three different parameters are varied. The results of this parameter study are presented in table 3.4.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Change</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Influence $V_1$ and $V_2$</td>
<td>Increase $V_1$</td>
<td>Higher wavenumber shift of the reflected wave and transmitted wave.</td>
</tr>
<tr>
<td>Influence $V_1$ and $V_2$</td>
<td>Decrease $V_2$</td>
<td>Lower wavenumber shift of the reflected wave and transmitted wave.</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>Decrease</td>
<td>Wavenumber shift of the reflected wave is the same, a higher wavenumber shift of the transmitted wave with an amplitude of approximately 1 [m].</td>
</tr>
<tr>
<td>Velocity difference</td>
<td>Increase</td>
<td>A larger wavenumber shift of the transmitted wave</td>
</tr>
</tbody>
</table>

Table 3.3: Results of an elastic rod with one inclusion that moves with a velocity $V_1$ (front) and a velocity $V_2$ (rear).

This simulation shows that it is possible to manipulate the center-frequency of the reflected and the transmitted wave and the amplitude of the transmitted wave is still the same as the input wave.
Chapter 4

Conclusions and recommendations

4.1 Conclusions

In this report the wave propagation through an elastic rod has been simulated. The simulation is done for three different cases. The first case that has been simulated, is an elastic rod with a static band gap structure. Subsequently, an elastic rod with a band gap structure which moves with a velocity \( V \) is considered. The last simulation was a special case where an elastic rod contains one inclusion which moves with a velocity \( V_1 \) at the front of the inclusion and with a velocity \( V_2 \) at the rear of the inclusion.

In the case of an elastic rod with a static band gap structure a parameter study is carried out. The results show that by changing the Young’s modulus the center-frequency of the band gap can be manipulated. When the Young’s modulus of the inclusions of the band gap structure is increased, the center-frequency of the band gap shifts to the higher center-frequency. Furthermore the maximum amplitude of the reflected wave is higher and the maximum transmitted amplitude of the wave will be lower. The number of inclusions has a large influence on the center-frequency of the band gap. More inclusions result in a deeper and wider band gap. The maximum transmitted amplitude of the reflected wave will increase when the number of inclusions increases. The influence of the inclusion width and the gap width are almost the same. For a small increase of the inclusion width, the center-frequency of the band gap shifts a little bit to the left, so other frequencies are transmitted. With these simulation settings it also shows that the maximum reflected wave will decrease for a smaller or a higher inclusion width or gap width. This simulation shows that with varying some parameters the amplitudes of the reflected waves and the transmitted waves can be manipulated. Also the main frequency content of the transmitted wave can be manipulated by changing some param-
eters. It is important to know that the input signal lies within the band gap, because otherwise the input signal transmits through the band gap structure.

In the second case the elastic rod with a band gap structure will be exposed to a velocity $V$. The velocity is increased from $V = 0 \ [m/s]$ to $V = 0.3 \ [m/s]$. From earlier studies the wavenumber shift of the reflected wave can be calculated. This calculation gives a good approximation of the wavenumber shift for a band gap structure which moves with a velocity $V$ within this range. This means that the wavenumber shift can be approximated for a velocity smaller than 0.3 \ [m/s]. For higher velocities it is not sure if the wavenumber shift can be approximated. Later studies should focus on the approximation of the wavenumber shift for higher velocities. Also this simulation shows that when the velocity is increased the maximum amplitude of the reflected wave will decrease. So by adding a velocity $V$ to the band gap structure it is possible to manipulate the wavenumber shift of the reflected waves and the maximum amplitude of the reflected waves. If a specific number of inclusions is exceeded, the number of inclusions does not influence the maximum amplitude of the reflected wave anymore. It is possible that for higher velocities the band gap structure can be seen as one inclusion, so there will be a reflected wave. Note that the simulation is done with 14 inclusions. It is not sure if the maximum reflected amplitude still shows the same decrease for increasing $V$. This is something that needs to be studied.

The last case an elastic rod with one inclusion which moves with a velocity $V_1$ at the front of the inclusion and a velocity $V_2$ at the rear of the inclusion is studied. The damping is turned on to reduce the high-frequency oscillations. A drawback of the extra damping is that it affects the amplitudes of the reflected and transmitted waves. This simulation shows that it is possible to get a wavenumber shift of the transmitted waves. It shows that a lower Young’s modulus results in a larger wavenumber shift of the transmitted wave. The wavenumber shift also depends on the velocity difference $\Delta V$. Furthermore a large $\Delta V$ shows a large wavenumber shift of the transmitted wave, while the reflected wavenumber shift is almost the same. The energy is not needed to be conserved for this case. Only a certain range of $V_1$ and $V_2$ is studied. Further study is recommended to see if the wavenumber shift of the transmitted wave is not enough.

4.2 Recommendations

The results that are presented here can be used for predicting what will happen with the input wave. Note that the results are approximations of the reality because a finite element model is used and it is solved by an explicit direct integration. Furthermore there is a lumped mass matrix used for fast calculation.
This can cause some loss of accuracy. It is recommended to use other mass matrices.

This report can be used for further study. An example of further research is to study what will happen with the wave propagation when an elastic rod with a band gap structure moves with different velocities. So each inclusion will get another velocity. Furthermore this new model can be extended so that each inclusion will gets a velocity $V_1$ at the front and a velocity $V_2$ at the rear of the inclusion or every inclusion has its own velocity $V_1$ and $V_2$. Also an elastic rod with an optimized band gap structure which moves with a velocity can be studied.
Appendix A

Derivation of the ordinary differential equation

Equation (2.1) is derived by using Newton’s second law and Hooke’s law. The forces that are working on a small particle of the structure are depicted in figure A.1. For an one-dimensional case the equation of Newton’s second law is

\[ F = m \cdot a \] (A.1)

In this equation \( a \) is the acceleration in \( [m/s^2] \). The acceleration \( a \) corresponds to the axial acceleration \( \dddot{u} = \partial^2 u / \partial t^2 \). The mass can be calculated by \( m = \partial x A \rho(x) \). Equation (A.1) can be written as

\[ F = \partial x A \rho(x) \frac{\partial^2 u}{\partial t^2} \] (A.2)

Figure A.1: The forces working on a small particle of the structure
Now the force $F$ on the left side of (A.2) needs to be calculated. By using the sign convention $F \rightarrow +$, the equation will change into

$$-\sigma A + (\sigma + \partial \sigma)A = \partial x A \rho(x) \frac{\partial^2 u}{\partial t^2}$$

$$-\sigma A + \sigma A + \partial \sigma A = \partial x A \rho(x) \frac{\partial^2 u}{\partial t^2}$$

$$\partial \sigma A = \partial x A \rho(x) \frac{\partial^2 u}{\partial t^2}$$

(A.3)

The $\sigma$ can be calculated by Hooke’s law, so

$$\sigma = E(x) \epsilon$$

(A.4)

In this equation $\epsilon$ is the dimensionless strain. The strain can be calculated by $\epsilon = \partial u/\partial x$. So (A.4) can be written as

$$\sigma = E(x) \frac{\partial u}{\partial x}$$

(A.5)

Subsequently, substituting (A.5) into (A.3) gives

$$\partial \left( E(x) \frac{\partial u}{\partial x} \right) A = \partial x A \rho(x) \frac{\partial^2 u}{\partial t^2}$$

(A.6)

The last step is to divide (A.6) by $\partial x A$. This results in the same equation as the physical problem of an elastic rod (see (2.1))

$$\frac{\partial}{\partial x} \left( E(x) \frac{\partial u}{\partial x} \right) = \rho(x) \frac{\partial^2 u}{\partial t^2}$$

(A.7)
Appendix B

Derivation of the boundary conditions

In this appendix the absorbing boundary equations and the damping coefficient of the viscous dampers are derived. Jensen [6] shows the derivations and these are presented below with some additional comments. A schematic representation of an elastic rod is depicted in figure B.1.

![Figure B.1: Elastic rod with absorbing boundary conditions](image)

The equation of motion for the elastic rod with constant material distribution is

\[(Eu')' = \rho \ddot{u}\]  \hspace{1cm} (B.1)

In this equation the prime denotes the partial derivative with respect to the variable \(x\).

For calculating the boundary condition of the left side of the elastic rod a free-body diagram is made. The free-body diagram is shown in figure B.2.

![Figure B.2: Free-body diagram of the left boundary condition](image)
This free-body diagram with the sign convention $F \to +$ yields
\[ C\dot{u} = f + AE \frac{\partial u}{\partial x} \quad (B.2) \]

So the force $f$ can be calculated by
\[ -AE \frac{\partial u}{\partial x} + C\dot{u} = f \quad (B.3) \]

Equation (B.1) can be solved as followed
\[
\begin{align*}
  u &= W e^{i\omega t} \\
  \dot{u} &= i\omega W e^{i\omega t} \\
  \ddot{u} &= i^2 \omega^2 W e^{i\omega t} = -\omega^2 W e^{i\omega t} 
\end{align*}
\quad (B.4)
\]

Substituting (B.4) in the equation of motion (B.1), yields
\[ (EW')' e^{i\omega t} = -\rho \omega^2 W e^{i\omega t} \quad (B.5) \]

Now dividing (B.5) by $e^{i\omega t}$ and rewriting the equation gives
\[ (EW')' + \rho \omega^2 W = 0 \quad (B.6) \]

The boundary condition can be written as
\[
\begin{align*}
  -AEW' + iC\omega W &= f \\
  -W' + i \left( \frac{C}{AE} \right) \omega W &= \left( \frac{f}{AE} \right) \quad \text{with} \quad \tilde{C} = \frac{C}{AE}, \quad \tilde{f} = \frac{f}{AE} \\
  -W' + i\tilde{C}\omega W &= \tilde{f} 
\end{align*}
\quad (B.7)
\]

The same method can be used for right boundary condition. In figure B.3 a free-body diagram is depicted.

![Free-body diagram](image)

Figure B.3: Free-body diagram of the right boundary condition

This free-body diagram with the sign convention $F \to +$ yields
\[ AEu' + C\dot{u} = 0 \quad (B.8) \]
Dividing (B.4) by $AE$ and substituting (B.8), yields

\[ AEW' + iC\omega W = 0 \]
\[ W' + i\tilde{C}\omega W = 0 \]  \hspace{1cm} (B.9)

Now the equation of motion (B.6) can be solved. First divide the equation by $E$ and introduce a variable $\gamma$, so the equation will be

\[ (EW')' + \rho\omega^2 W = 0 \]
\[ W'' + \frac{\rho\omega^2}{E} W = 0 \hspace{0.5cm} \text{with } \gamma^2 = \frac{\rho\omega^2}{E} \text{ so } \gamma = \sqrt{\frac{\rho}{E}} \omega \text{ yields} \]
\[ W'' + \gamma^2 W = 0 \]  \hspace{1cm} (B.10)

This equation can be solved with

\[ W = A e^{-i\gamma x} + B e^{i\gamma x} \]
\[ W' = -i\gamma A e^{-i\gamma x} + i\gamma B e^{i\gamma x} \]
\[ W'' = -\gamma^2 A e^{-i\gamma x} - \gamma^2 B e^{i\gamma x} \]  \hspace{1cm} (B.11)

where $x \in [0; L]$. For the left boundary condition $x = 0$ and for the right boundary condition $x = L$. Substitute (B.11) in (B.7) for the left side ($x = 0$) and in (B.9) for the right side ($x = L$). This yields for the left side

\[ i\gamma A e^{-i\gamma 0} - i\gamma B e^{i\gamma 0} + \omega \tilde{C}(A e^{-i\gamma 0} + B e^{i\gamma 0}) = \tilde{f} \]  \hspace{1cm} (B.12)

and this yields for the right side

\[ -i\gamma A e^{-i\gamma L} + i\gamma B e^{i\gamma L} + \omega \tilde{C}(A e^{-i\gamma L} + B e^{i\gamma L}) = 0 \]  \hspace{1cm} (B.13)

Require $B = 0$ and $A = W_0$ gives for the right side

\[ -i\gamma W_0 e^{-i\gamma L} + i\omega \tilde{C}W_0 e^{-i\gamma L} = 0 \]
\[ -i\gamma W_0 + i\omega \tilde{C}W_0 = 0 \]  \hspace{1cm} (B.14)

Now $\tilde{C}$ can be calculated by substitution $\gamma$ of (B.10), this yields

\[ \tilde{C} = \frac{\gamma}{\omega} = \sqrt{\frac{\rho}{E}} \omega = \sqrt{\frac{\rho}{E}} \]  \hspace{1cm} (B.15)

The damping coefficient can be calculated with (B.7) and gives

\[ C = AE\tilde{C} = AE \sqrt{\frac{\rho}{E}} = A\sqrt{\rho E} \]  \hspace{1cm} (B.16)
The original boundary condition (B.3) for the left side can be written as

\[-AE \frac{\partial u}{\partial x} + A \sqrt{\rho E} \frac{\partial u}{\partial t} = f \text{ divide by } AE\]

\[-\frac{\partial u}{\partial x} + \sqrt{\frac{\rho}{E}} \frac{\partial u}{\partial t} = \frac{f}{AE} \text{ with } c = \sqrt{\frac{E}{\rho}}\]

\[-\frac{\partial u}{\partial x} + \frac{1}{c} \frac{\partial u}{\partial t} = \frac{f}{AE}\]  
(B.17)

This equation corresponds to the boundary condition at the left end that is shown in (2.2). The original boundary condition (B.8) for the right side can be written as

\[AE \frac{\partial u}{\partial x} + A \sqrt{\rho E} \frac{\partial u}{\partial t} = 0 \text{ divide by } AE\]

\[\frac{\partial u}{\partial x} + \sqrt{\frac{\rho}{E}} \frac{\partial u}{\partial t} = 0 \text{ with } c = \sqrt{\frac{E}{\rho}}\]

\[\frac{\partial u}{\partial x} + \frac{1}{c} \frac{\partial u}{\partial t} = 0\]  
(B.18)

Equation (B.18) corresponds to the boundary condition at the right end that is shown in (2.3).
Appendix C

Element mass matrices

In Cook et al. [3] some different element mass matrices are presented. In this appendix the two other element mass matrices are described. The other two element mass matrices are the consistent element mass matrix and the combination element mass matrix are given.

The consistent element mass matrix is

\[
[m] = \frac{m}{6} \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}
\]  \hspace{1cm} (C.1)

where \(m\) is the element mass. The element mass can be calculated by \(m = \rho AL\) in [kg], where \(\rho\) is the mass density in [kg/m^2], \(A\) is the cross-sectional area in [m^2] and \(L\) is the element length in [m]. By using the consistent element mass matrix (C.1) the global mass matrix is constructed. The form of the global mass matrix for an elastic rod with the same material distribution is

\[
M = \frac{\rho AL}{6} \begin{bmatrix}
2 & 1 & 0 & \cdots & 0 \\
1 & 4 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 4 & 1 \\
0 & \cdots & 0 & 1 & 2
\end{bmatrix}
\]  \hspace{1cm} (C.2)

The combination element mass matrix is a combination of the element lumped mass matrix and the consistent element mass matrix. In the most cases a combination mass matrix is more accurate. To calculate the element mass matrix a factor \(\beta\) between \(0 \leq \beta \leq 1\) is used.
The combination element mass matrix is calculated by

\[
\begin{bmatrix} m \end{bmatrix} = (1 - \beta) \cdot \text{lumped} + \beta \cdot \text{consistent}
\]

\[
\begin{bmatrix} m \end{bmatrix} = (1 - \beta) \frac{m}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \beta \frac{m}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}
\]

\[
\begin{bmatrix} m \end{bmatrix} = \frac{m}{6} \begin{bmatrix} 3 - \beta & \beta \\ \beta & 3 - \beta \end{bmatrix}
\]

(C.3)

The average element mass matrix is obtained with \( \beta = 0.5 \). The element mass matrix will be

\[
\begin{bmatrix} m \end{bmatrix} = \frac{m}{12} \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}
\]

(C.4)

The form of the global mass matrix for an elastic rod with the same material distribution is

\[
\begin{bmatrix} M \end{bmatrix} = \frac{\rho AL}{12} \begin{bmatrix} 5 & 1 & 0 & \cdots & 0 \\ 1 & 10 & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & \ddots & 0 \\ \vdots & \cdots & \cdots & \cdots & 10 \\ 0 & \cdots & 0 & 1 & 5 \end{bmatrix}
\]

(C.5)
Appendix D

Equilibrium equations for a two-node rod element

In Cook et al. [3] the calculation of the equilibrium equations are presented. The calculation of the equilibrium equations is done using the free-body diagram (see figure 2.3). By making use of Newton’s first law and sign convention \( \sum F = 0 \) the equilibrium equation for node 1 and node 2 can be calculated with

\[
\begin{align*}
F_1 + A\sigma &= 0 \\
F_2 - A\sigma &= 0 \\
\sigma &= E\epsilon \\
\epsilon &= \frac{u_2 - u_1}{L}
\end{align*}
\]  

(D.1)

Rewriting (D.1), yields

\[
\frac{AE}{L}(u_1 - u_2) = F_1
\]

or

\[
\begin{bmatrix}
\frac{AE}{L} & -\frac{AE}{L} \\
-\frac{AE}{L} & \frac{AE}{L}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
=
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
\]

(D.2)

\[
\frac{AE}{L}(u_2 - u_1) = F_2
\]

So the element stiffness matrix is

\[
[k] = \begin{bmatrix}
k & -k \\
-k & k
\end{bmatrix}
\]

with \( k = \frac{AE}{L} \)  

(D.3)
Appendix E

Classical finite differences

Cook et al. [3] explains the classical finite difference method. The idea by using the central finite difference is to approximate the velocity and acceleration at step $n$ for each time step $\Delta t$. By using the Taylor series expansions of $u_{n+1}$ and $u_{n-1}$ (see equation E.1) the velocity and the acceleration can be obtained.

\begin{align*}
    u_{n+1} &= u_n + \Delta t \dot{u}_n + \frac{(\Delta t)^2}{2} \ddot{u}_n + \frac{(\Delta t)^3}{6} \dddot{u}_n + \ldots \quad (E.1a) \\
    u_{n-1} &= u_n - \Delta t \dot{u}_n + \frac{(\Delta t)^2}{2} \ddot{u}_n - \frac{(\Delta t)^3}{6} \dddot{u}_n + \ldots \quad (E.1b)
\end{align*}

The velocity $\dot{u}_n$ can be calculated by subtracting (E.1b) from (E.1a), this results in

$$u_{n+1} - u_{n-1} = 2\Delta t \dot{u}_n + \frac{2(\Delta t)^3}{6} \dddot{u}_n + \ldots \quad (E.2)$$

Because $\Delta t$ is small, so $(\Delta t)^3$ is smaller and almost equal to zero. In this case the term $2(\Delta t)^3/6 \dddot{u}_n$ can be neglected. Rewriting this equation yields

$$\dot{u}_n = \frac{1}{2\Delta t} (u_{n+1} - u_{n-1}) \quad (E.3)$$

The acceleration $\ddot{u}_n$ can be calculated by adding (E.1b) with (E.1a) and this gives

$$u_{n+1} + u_{n-1} = 2u_n + \frac{2(\Delta t)^2}{2} \ddot{u}_n + \ldots \quad (E.4)$$

The higher order terms of $\Delta t$ can be neglected. Rewriting (E.4), yields

$$\ddot{u}_n = \frac{1}{(\Delta t)^2} (u_{n+1} - 2u_n + u_{n-1}) \quad (E.5)$$
Substitute (E.3) and (E.5) into the mathematical model (2.18) provides

\[
\left( \frac{1}{(\Delta t)^2} M + \frac{1}{2\Delta t} C \right) u_{n+1} = F_n - K_n u_n + \frac{2}{(\Delta t)^2} M u_n
\]

\[- \left( \frac{1}{(\Delta t)^2} M - \frac{1}{2\Delta t} C \right) u_{n-1} \quad (E.6)
\]

This equation can be simplified when the damping terms of (E.6) are gathered together. The damping can be approximated by

\[
\frac{1}{2\Delta t} C (u_{n-1} - u_{n+1}) \approx \frac{1}{\Delta t} C (u_{n+1} - u_n) \quad (E.7)
\]

Substitution with (E.6), yields

\[
\frac{1}{(\Delta t)^2} M u_{n+1} \approx F_n - K_n u_n + \left( \frac{2}{(\Delta t)^2} M - \frac{1}{\Delta t} C \right) u_n
\]

\[- \left( \frac{1}{(\Delta t)^2} M - \frac{1}{\Delta t} C \right) u_{n-1} \quad (E.8)
\]

This is the same equation as (2.20).
References


