Coupled oscillators in near-periodic elastic structures

S.W.H. Versteeg

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Traineeship report

Coach(es): J.S. Jensen, Assoc.prof. Ph.D.
J.J. Thomsen, Assoc.prof. dr.techn.

Supervisor: Prof. dr. H. Nijmeijer

Technische Universiteit Eindhoven
Department Mechanical Engineering
Dynamics and Control Technology Group

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Abstract

A setup of multiple coupled non-linear pendula can exhibit dynamics like synchronization and localization. These phenomena can cause strong vibrations, which can be undesired and harmful to a structure. To investigate these group dynamics an experimental setup with swinging pendula is used. This setup consists of multiple piano metronomes coupled by a common base. In this work the influence of natural frequency differences on group dynamics is investigated.
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Chapter 1

Introduction

In many structures consisting of a series of semi-identical non-linear substructures, undesired group-dynamics can occur like synchronization and localization. For interacting systems synchronization means that the frequencies of the individual systems concur over time, while the phase differences evolve to certain constant values (definitions for synchronization can be found in [1]). In-phase synchronization can cause strong vibrations, which can be undesired and harmful to the structure. Localization is the stronger occurrence of a certain mode in a part of a structure, having a slightly different eigenfrequency than the rest of the substructures (more on localization phenomena can be found in [2]). To investigate synchronization related phenomena we go back to the most basic practical setup using swinging pendula. A pendulum is one of the simplest non-linear structures, which make them suitable for the realization of an experimental setup. In this work the influence of natural frequency differences on group dynamics is investigated.

Since the discovery of the phenomenon of spontaneous synchronization by C. Huygens in the 16th century [3], a lot of research has been conducted in the field of synchronization of coupled oscillators. In general a rough distinction can be made between research on systems like the Huygens setup, consisting of two coupled pendula and research done on networks of mutually coupled quasi-periodic oscillators of arbitrary kind.

In Bennett et al. [4] Huygens’ original results have been re-examined and reproduced. Using very similar but not identical clocks, anti-phase synchronization and beating death behavior were experimentally observed for different coupling strengths. Pantaleone [5] used a slightly different setup of two (and also more) metronomes placed on a freely moving base. He observed both in-phase and anti-phase synchronization. Inspired by the work of Bennett and Pantaleone, Oud et al. [6] conducted further research on a Huygens setup consisting of two pendula coupled by a weak spring, which led to more insight on conditions for synchronization. More work on this Huygens setup is done by Fradkov et al. [7], where dependency of pendulum phases on system parameters and initial conditions is studied.

An important contribution to the modelling of networks of large numbers of oscillators was made by Kuramoto [8], [9], inspired by the work of Winfree [10]. The ‘Kuramoto model’ exists of a network of coupled phase-oscillators, in which phase modulation is the main
cause for the group dynamics. This model can show a wide variety of dynamics, although amplitude modulation is not accounted for. Matthews et. al. [11] generalized the phase model by allowing the oscillators to undergo amplitude variations as well, which allows for more types of collective behavior.

Inspired by all the work done on two pendula systems, in this work the dynamic behavior of a setup consisting of more than two coupled pendula is investigated. In many practical situations more than two semi-identical nonlinear subsystems are interacting. Therefore it would be valuable to gain more insight in the synchronization dynamics of a setup consisting of multiple coupled pendula. The work of Pantaleone [5] will be used as a lead in many parts of this work. The used setup consists of a light wooden board on which up to five piano metronomes, which act as pendula, can be placed. The wooden board is free to translate in one direction, thus acting as a horizontal coupling for the pendula. The pendula are set at different eigenfrequencies, which are chosen to be non-randomly distributed according to a uniform distribution. The dynamic behavior of the system is studied for different distributions of the pendulum natural frequencies: the frequency distribution average as well as the distribution width will be varied. Special attention will be for the conditions under which in-phase synchronization occurs.

This report is organized as follows. In Section 2 first the dynamic model of a single pendulum is posed and second this model is extended to the multiple coupled pendula setup. To make predictions on the behavior of the system simulations are done using a numerical analysis tool, which is based on this model. In Section 3 the experimental setup is treated, which is used to test the predictions from the numerical analysis tool in Section 2. In Section 4 the results from the simulations and the experiments are discussed. In Section 5, a summary of the presented work and conclusions and recommendations are made.
Chapter 2

Modelling and simulations

In this section a model is derived for the experimental setup of the multiple coupled pendula system, which will be treated in Section 3. The model is used to do simulations, which can be used to predict the occurrence of in-phase synchronization for arbitrary settings.

2.1 Model for two coupled pendula

![Diagram of 2-pendula setup](image)

Figure 2.1: Model of the 2-pendula setup used for deriving equations of motion

In the first part of this chapter a model is proposed for a setup consisting of multiple
pendula coupled by a base, which has one translational degree of freedom in the horizontal direction. For a better understanding of the model that will be derived for this setup, first model equations for a similar setup with only two coupled pendula are derived.

### 2.1.1 Lagrange’s equations in absence of friction

Generalized coordinates \( \mathbf{q}^T = [x \ \theta_1 \ \theta_2] \) are used, where \( \theta_i \) is the angle of each pendulum with the vertical, \( x \) is the displacement of the base (with \( x = 0 \) for the system at rest).

If friction is omitted for the sake of simplicity, the equations of motion for the double pendulum system shown in Fig. 2.1 can be derived using Lagrange’s equations as follows. The position vectors \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) of the center of gravity of each pendulum are expressed in the chosen Cartesian coordinate frame using the generalized coordinates

\[
\mathbf{p}_1 = \begin{bmatrix} x + a_1 + r_1 \sin \theta_1 \\ -r_1 \cos \theta_1 \end{bmatrix}, \quad (2.1a)
\]

\[
\mathbf{p}_2 = \begin{bmatrix} x + a_2 + r_2 \sin \theta_2 \\ -r_2 \cos \theta_2 \end{bmatrix}, \quad (2.1b)
\]

from which their velocities can be derived as

\[
\dot{\mathbf{p}}_1 = \begin{bmatrix} \dot{x} + r_1 \cos \theta_1 \dot{\theta}_1 \\ r_1 \sin \theta_1 \dot{\theta}_1 \end{bmatrix}, \quad (2.2a)
\]

\[
\dot{\mathbf{p}}_2 = \begin{bmatrix} \dot{x} + r_2 \cos \theta_2 \dot{\theta}_2 \\ r_2 \sin \theta_2 \dot{\theta}_2 \end{bmatrix}. \quad (2.2b)
\]

This allows the kinetic energy \( T \) of the system to be written as

\[
T = \frac{1}{2} \dot{\mathbf{p}}_1^T m_1 \dot{\mathbf{p}}_1 + \frac{1}{2} \dot{\mathbf{p}}_2^T m_2 \dot{\mathbf{p}}_2 + \frac{1}{2} I_{c1} \dot{\theta}_1^2 + \frac{1}{2} I_{c2} \dot{\theta}_2^2 + \frac{1}{2} M \dot{x}^2, \quad (2.3a)
\]

\[
= \frac{1}{2} (M + m_1 + m_2) \dot{x}^2 + \frac{1}{2} I_{c1} \dot{\theta}_1^2 + \frac{1}{2} I_{c2} \dot{\theta}_2^2 + \frac{1}{2} m_1 \left(2 r_1 \cos \theta_1 \dot{x} \dot{\theta}_1 + r_1^2 \dot{\theta}_1^2\right) + \frac{1}{2} m_2 \left(2 r_2 \cos \theta_2 \dot{x} \dot{\theta}_2 + r_2^2 \dot{\theta}_2^2\right), \quad (2.3b)
\]

where \( M \) is the total base mass, \( m_i \) and \( I_{ci} \) are the mass and the central moment of inertia of each pendulum, \( r_i \) is the distance of the center of mass from the pivot point of each pendulum and \( g \) is the gravitational acceleration. This result can be used to derive

\[
T^T_{\mathbf{q}} = \begin{bmatrix} (M + m_1 + m_2) \dot{x} + m_1 r_1 \cos \theta_1 \dot{\theta}_1 + m_2 r_2 \cos \theta_2 \dot{\theta}_2 \\ m_1 r_1 \cos \theta_1 \dot{x} + (m_1 r_1^2 + I_{c1}) \dot{\theta}_1 \\ m_2 r_2 \cos \theta_2 \dot{x} + (m_2 r_2^2 + I_{c2}) \dot{\theta}_2 \end{bmatrix}, \quad (2.4)
\]

and
\[
\frac{d}{dt} (T_q)^T = \begin{bmatrix}
(M + m_1 + m_2) \ddot{x} + m_1 r_1 \left( -\sin \theta_1 \dot{\theta}_1^2 + \cos \theta_1 \ddot{\theta}_1 \right) + m_2 r_2 \left( -\sin \theta_2 \dot{\theta}_2^2 + \cos \theta_2 \ddot{\theta}_2 \right) \\
m_1 r_1 \left( -\sin \theta_1 \dot{\theta}_1 + \cos \theta_1 \ddot{x} \right) + (m_1 r_1^2 + I_{c1}) \dot{\theta}_1 \\
m_2 r_2 \left( -\sin \theta_2 \dot{\theta}_2 + \cos \theta_2 \ddot{x} \right) + (m_2 r_2^2 + I_{c2}) \dot{\theta}_2
\end{bmatrix},
\]

and also
\[
T_{T,q}^T = \begin{bmatrix}
0 \\
-m_1 r_1 \sin \theta_1 \dot{x} \dot{\theta}_1 \\
-m_2 r_2 \sin \theta_2 \dot{x} \dot{\theta}_2
\end{bmatrix}.
\]

Furthermore the potential energy \( V \) can be written as:
\[
V = -m_1 r_1 g \cos \theta_1 - m_2 r_2 g \cos \theta_2,
\]
leading to
\[
V_{T,q}^T = \begin{bmatrix}
0 \\
m_1 r_1 g \sin \theta_1 \\
m_2 r_2 g \sin \theta_2
\end{bmatrix}.
\]

Because non-conservative forces are absent, the column of non-conservative forces is
\[
Q^{nc} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
\]

Substitution of Eq. (2.5), Eq. (2.6) and Eq. (2.8) in Lagrange's equation
\[
\left[ \frac{d}{dt} (T_q) - T_q + V_q \right]^T = Q^{nc},
\]
leads to the equations of motion:
\[
(M + m_1 + m_2) \ddot{x} + m_1 r_1 \left( -\sin \theta_1 \dot{\theta}_1^2 + \cos \theta_1 \ddot{\theta}_1 \right) + m_2 r_2 \left( -\sin \theta_2 \dot{\theta}_2^2 + \cos \theta_2 \ddot{\theta}_2 \right) = 0,
\]
\[
I_1 \ddot{\theta}_1 + m_1 r_1 g \sin \theta_1 + m_1 r_1 \cos \theta_1 \ddot{x} = 0,
\]
\[
I_2 \ddot{\theta}_2 + m_2 r_2 g \sin \theta_2 + m_2 r_2 \cos \theta_2 \ddot{x} = 0,
\]
where the total inertia around the pivot point, \( m_i r_i^2 + I_{ci} \), is replaced by \( I_i \).

Equation (2.11a) can be written slightly different as

\[
\ddot{x} = \frac{m_1 r_1 \left( -\sin \theta_1 \dot{\theta}_1^2 + \cos \theta_1 \ddot{\theta}_1 \right) + m_2 r_2 \left( -\sin \theta_2 \dot{\theta}_2^2 + \cos \theta_2 \ddot{\theta}_2 \right)}{M + m_1 + m_2}.
\]  

(2.12)

Double integration of Eq. (2.12) (with zero integration constants) would lead to

\[
x = \frac{-m_1 r_1 \sin \theta_1 - m_2 r_2 \sin \theta_2}{M + m_1 + m_2},
\]  

(2.13)

which shows that the displacement \( x \) can be expressed in \( \theta_1 \) and \( \theta_2 \). In the ideal frictionless case there are no non-conservative forces from the surroundings acting on the base and metronomes. Thus solely the angles of the metronomes dictate the displacement of the base.

Apart from the missing friction and escapement model, which is treated in the following section, the form of Eq. (2.11) (and also Eq. (2.13)) is almost similar to Eqs. (1) and (5) from [5]. The difference is that each metronome is assigned an individual mass \( m_i \), inertia \( I_i \) and distance \( r_i \).

2.1.2 Addition of essential friction model

In Section 2.1.1 model equations for two coupled frictionless pendula are derived. In reality there is friction present in each pendulum. This would cause the pendulum’s motion to die out if there is no compensation for this energy dissipation. In the piano metronome used in the experiments this problem is solved by adding a wind-up mechanism: each time the pendulum swings beyond a certain angle (so twice each period), this mechanism gives a small kick to the pendulum to keep it going (the experimental setup is treated in Section 3.1.1). Also from a physical modelling point of view it is essential to add friction: without friction there is no dissipation of energy from transients, this will cause the system never to reach a steady state, because the transients cannot die out.

The metronome escapement mechanism can be modelled in many ways. In Section 2.5 the influence of this model on in-phase synchronization dynamics is studied briefly for the two pendula setup. From those results the assumption is made that this model is not of large influence on the in-phase synchronization dynamics of the coupled pendula setup (as mentioned in Section 2.5, this might not be the case for general dynamic behavior of the setup). Thus a simple model is chosen, which models both energy dissipation and energy resupply: a quadratic term is added to the equations of motion as in [5] (such a term is often
called a van der Pol term). Hence the new equations of motion become:

\[
\ddot{\theta}_1 + \frac{m_1 r_1 g}{I_1} \sin \theta_1 + \epsilon \left[ \left( \frac{\theta_1}{\theta_0} \right)^2 - 1 \right] \dot{\theta}_1 + \frac{m_1 r_1}{I_1} \cos \theta_1 \ddot{x} = 0, \tag{2.14a}
\]

\[
\ddot{\theta}_2 + \frac{m_2 r_2 g}{I_2} \sin \theta_2 + \epsilon \left[ \left( \frac{\theta_2}{\theta_0} \right)^2 - 1 \right] \dot{\theta}_2 + \frac{m_2 r_2}{I_2} \cos \theta_2 \ddot{x} = 0, \tag{2.14b}
\]

where \( \epsilon \) is a characteristic friction parameter, \( \theta_0 \) is a characteristic amplitude and \( \ddot{x} \) can again be obtained from Eq. (2.12). For small \( \epsilon \) and \( \theta_0 \) the addition of the van der Pol term will produce stable oscillations.

It should be noted that the friction of the base with the soda cans and the soda cans with the ground is neglected, because it is considered to be rather small.

### 2.2 Model for multiple coupled pendula

Equation (2.14) can easily be expanded for a system of \( n \) pendula in a straightforward systematic way. Figure 2.2 shows an illustration of such a system for \( n = 5 \).

#### 2.2.1 Derivation of scaled, dimensionless equations

A scaled, dimensionless time variable \( \tau = \omega_1 t \) is introduced and the single pendulum natural frequency for small angles (linear case)

\[
\omega_i^2 = \frac{m_i r_i g}{I_i}, \tag{2.15}
\]

is used. Furthermore for pendulum \( i \) a dimensionless inertia parameter \( \zeta_i \)

\[
\zeta_i = \frac{m_i r_i \omega_1^2}{g \left( M + \sum_{k=1}^{n} m_k \right)}, \tag{2.16}
\]

is defined and also a dimensionless natural eigenfrequency parameter \( \alpha_i \)

\[
\alpha_i = \frac{\omega_i}{\omega_1}, \tag{2.17}
\]

which obviously leads to \( \alpha_1 = 1 \). From now on the notation \( \dot{\theta}_i \) will be used to indicate \( \frac{d\theta_i}{d\tau} \) instead of \( \frac{d\theta_i}{dt} \). This allows for the differential equations to be written in the following
dimensionless form:

\[
\ddot{\theta}_i + \alpha_i^2 \sin \theta_i + \mu \left[ \left( \frac{\theta_i}{\theta_0} \right)^2 - 1 \right] \dot{\theta}_i + \alpha_i^2 \cos \theta_i \ddot{x}^* = 0, \quad i = 1, n, \tag{2.18a}
\]

\[
\ddot{x}^* = \sum_{j=1}^{n} \zeta_j \left( \sin \theta_j \dot{\theta}_j^2 - \cos \theta_j \ddot{\theta}_j \right), \tag{2.18b}
\]

where \( \ddot{x}^* \) is the scaled base displacement and \( \mu = \frac{\epsilon}{\omega_1} \) is the scaled parameter for the van der Pol term.

![Diagram of the n-pendula setup](image)

Figure 2.2: Model of the \( n \)-pendula setup \( (n = 5) \).

The individual coupling strength, which determines the influence of pendulum \( j \) on pendulum \( i \), can be written as

\[
\beta_{ij} = \alpha_i^2 \zeta_j. \tag{2.19}
\]

An important observation about the equations of motion, Eq. (2.18), can be made. Because pendulum \( i \) is coupled to the base movement (and thus to the other pendula) solely by \( \alpha_i^2 \cos \theta_i \ddot{x}^* \), pendulum \( i \) cannot distinguish itself in its behavior from the other pendula by having a different inertia \( \zeta_i \). In other words, contrary to what one might expect intuitively, the swinging amplitude of pendulum \( i \) for instance is not directly dependent on its inertia \( \zeta_i \). However, because the base movement itself is caused by all pendulum’s \( \zeta \), the change of one \( \zeta_i \) of a particular pendulum \( i \) can change the overall system behavior in theory. This observation is not further investigated experimentally or numerically.
### 2.2.2 From second order to first order differential equations

The system resulting from Eq. (2.18) is a system of $n + 1$ second order equations of motion. For numerical simulations it is convenient to rewrite this system in first order form. Because the state $x^*$ is not required to calculate $\ddot{x}^*$ and because $x^*$ can actually be calculated using an expanded version of Eq. (2.13) if desired, $x^*$ is discarded in the dynamics, which leads to the following system of $2n + 1$ equations:

\begin{align*}
\dot{y}_{2i-1} &= y_{2i}, \\
\dot{y}_{2i} &= -\alpha_i^2 \cos y_{2i-1} y_{2n+1} - \alpha_i^2 \sin y_{2i-1} - \mu \left( \frac{y_{2i-1}}{\theta_0} \right)^2 - 1 \right) y_{2i}, \quad i = 1, n, \\
\dot{y}_{2n+1} + \sum_{j=1}^{n} \zeta_j \cos y_{2j-1} \dot{y}_{2j} &= \sum_{j=1}^{n} \zeta_j \sin y_{2j-1} y_{2j}^2, \\
\end{align*}

with states $y^T = [\theta_1 \ \dot{\theta}_1 \ \theta_2 \ \dot{\theta}_2 \ \cdots \ \theta_n \ \dot{\theta}_n \ \dot{x}^*]$. This result will be used for the numerical simulation tool in Section 2.3.

A benefit of this split-up is that both displacement and velocity states are calculated, which allows for easy extraction of phase and amplitude estimations (more on this in Section 2.6).
2.3 Numerical implementation of the model

Predictions for a particular solution of a particular \( n \)-pendula system can be made by solving the equations of motion, Eq. (2.20), numerically. For this Matlab’s solver \texttt{ode45} is used with relative tolerance \( 10^{-4} \). Using a short Matlab script, a mass matrix \( M \) and function vector \( f \) are systematically generated for the \( n \) pendula system of equations \( M\dot{y} = f \). Hence the states \( y \) (as defined in Section 2.2.2), can be solved numerically.

For the terms in Eq. (2.20) including first order derivatives, \( \dot{\theta}_i \), the choice has to be made to put them in the left hand side or the right hand side. The terms can be put in either the mass matrix \( M \) or in the vector function \( f \). This is a result of the transformation of the set of second order systems into first order form in Section 2.2.2, which defines \( y'_{2i-1} = y_{2i} = \dot{\theta}_i \).

Both solving methods have been compared with relative tolerance \( 10^{-6} \), the relative error after simulating for 10,000 seconds (in dimensionfull time \( t \)) is of order \( 10^{-10} \), from which can be concluded that there is no significant difference between the results of both methods. For maximum sparseness of matrix \( M \) the terms are put in \( f \).

2.4 Test of numerical tool

To test the implementation of the numerical tool in matlab, numerical results based on Eq. (2.20) are compared to results from the timeplot shown in Fig. 2 from [5].

Because the trajectories of the pendula in the phase plane are close to circles around the origin, an estimate \( \psi \) of the phase difference between two pendula is calculated as

\[
\psi = \phi_2 - \phi_1, \quad (2.21)
\]

where \( \phi_i \) is the phase of pendulum \( i \) at a particular time instance. The relation between the three parameters \( \phi_1, \phi_2 \) and \( \psi \) is shown graphically in Fig. 2.3.

If the variation of \( \psi \) over each period is assumed small, the following holds for the relative time lag \( \delta t/t \):

\[
\frac{\delta t}{t} = \frac{\delta \tau}{\tau} \approx \frac{\psi}{2\pi}. \quad (2.22)
\]

The numerical \( n \)-pendula model from Section 2.3 is used with \( n = 2 \) and \( \theta_0 = 0.39 \) rad and \( \mu = 0.010 \) as in [5]. The parameters \( \alpha_i \) and \( \zeta_i \) are calculated from [5] as well and they are \( \alpha_1 = \sqrt{1.001}, \alpha_2 = \sqrt{0.999} \) and \( \zeta_1 \approx \zeta_2 = 0.011 \). Initial conditions are chosen to best fit the plot shown in Fig. 2 from [5] and are \( \dot{\theta}_1 = 0.78, \dot{\theta}_1 = 0, \dot{\theta}_2 = -0.78, \dot{\theta}_2 = -0.05 \) and \( \ddot{x}^* = 0 \).

In Fig. 2.4 both the original plot in Fig. 2 from [5] and the reproduced plot are shown. As can be observed the results are very similar and thus the numerical model is assumed to be viable.
Figure 2.3: Calculation of phase difference $\psi$ for the two pendulum system ($n = 2$). Phases are defined clockwise, because cycles in the phase plane are clockwise.

Figure 2.4: Plot of the relative time lag $\delta t/t$ versus time $t$ for the two pendulum system. The dots show the numerical solution in Fig 2. from [5] and the solid line shows a numerical solution of the $n$-pendula model from Eq. (2.18) using $n = 2$ and similar initial conditions.
2.5 Energy dissipation and energy resupply model

To investigate if the energy dissipation and energy resupply model (which is of the general form $f(\theta_i) \dot{\theta}_i$ in Eq. (2.18a)) has significant influence on the synchronization dynamics of the multiple pendula model, an alternative energy dissipation and energy resupply model is tested numerically. For this purpose the smooth continuous quadratic van der Pol function

$$f(\theta_i) = \mu \left( \left( \frac{\theta_i}{\theta_0} \right)^2 - 1 \right),$$

is replaced with a completely different, more spiky, Gaussian pulse function:

$$f(\theta_i) = \mu \left[ 1 - b e^{-\left( \frac{\theta_i}{c} \right)^2} \right].$$

By comparing numerical results on in-phase synchronization behavior from both models for $\theta_0 = 0.39$ rad, for different initial conditions, parameter values for $b$ and $c$ are found that yield nearly the same results: after some tuning, $b = 25$ and $c = 0.03$ are chosen for the Gaussian pulse to best match the numerical results using the van der Pol function.

In Fig. 2.5 the two different terms for $f(\theta)$ are shown. Although this replacement term clearly produces totally different dissipation and driving forces, simulations with this model produce almost the same results concerning in-phase synchronization dynamics. Figure 2.6 shows the result for the same initial conditions used in Section 2.4, the amplitudes are calculated as shown in Fig. 2.3. More simulations for different initial conditions, using the same parameter values for $b$ and $c$ confirm that this alternative model produces similar results. From this observation can be concluded that the way the energy dissipation and energy resupply of the pendula is modelled is not a dominating factor for the in-phase synchronization dynamics of the model for the multiple coupled pendula system from Section 2.2.1. The van der Pol term is chosen to be used to model the energy dissipation and energy resupply. Besides the fact that this smooth model makes numerical simulations computationally less straining, it is also tractable. This makes analytical work on the model equations easier.

It should be stressed that the model is not tested thoroughly and therefore it is not possible to assume that it covers all of the dynamics of the experimental setup.

2.6 Estimating amplitude and phase

A phase difference like $\psi$ from Eq. (2.21) is not a suitable indicator for phase-lock synchronization if $n > 2$. To show the occurrence of phase-lock synchronization for any $n$ a similar...
Figure 2.5: Two different functions $f(\theta)$ to model energy dissipation and energy resupply versus pendulum angle $\theta$. The solid line shows the van der Pol function from Eq. (2.23), the dashed line shows the completely different Gaussian pulse from Eq. (2.24).

Figure 2.6: Relative time $\delta t/t$ lag versus time $t$ using two different models for energy dissipation and energy resupply. The solid line shows a numerical solution using the smooth van der Pol term, the dashed line shows a numerical solution for the same initial conditions but now with the Gaussian pulse term instead.
method is proposed. A rotating reference frame is defined by the mean of all pendulum phases
\[
\overline{\phi} = \frac{1}{n} \sum_{i=1}^{n} \phi_i, \tag{2.25}
\]
which can be subtracted from each phase to end up with a relative phase \(\Delta \phi_i\) for each pendulum
\[
\Delta \phi_i = \phi_i - \overline{\phi}, \tag{2.26}
\]
and using the same assumption used for Eq. (2.22) the new relative time lags are defined:
\[
\frac{\Delta \tau_i}{\tau} \approx \frac{\Delta \phi_i}{2\pi}, \tag{2.27}
\]
which can be used as indicators for phase-lock synchronization (for the two pendula setup phases \(\phi_1\) and \(\phi_2\) are drawn in Fig. 2.3). In addition this definition makes it possible to observe an estimate for the steady-state phase differences.

### 2.7 Simulation with 3 pendula

Figure 2.7 shows a simulation for \(n = 3\) pendula running at \(\omega_{pen} = 209, 208, 207\) BPM (beats per minute) and with initial conditions \(\theta_1 = 0.78, \dot{\theta}_1 = 0.2, \theta_2 = -0.78, \dot{\theta}_2 = -0.05, \theta_3 = 0.6, \dot{\theta}_3 = -0.2\). This plot clearly shows that inspection of the actual states \(\theta_i\) and \(\dot{\theta}_i\) is cumbersome because of the large difference in timescales between the slow and fast behavior (it is hardly possible to distinguish \(\theta_1, \theta_2\) and \(\theta_3\)). It is more convenient to interpret the estimated amplitudes \(A_i\) and relative time lags \(\Delta \tau_i/\tau\), for they can clearly show which interesting slow timescale effects unfold.

In this example the pendula end up in a steady state of in-phase phase-lock synchronization: all relative phases \(\Delta \tau_i/\tau\) seem to evolve to zero. A closer inspection in Fig. 2.8 shows that after a long time \(\Delta \tau_i/\tau\) and \(A_i\) end up around steady-state values. The fast pendulum 1 shows the largest amplitude and is ahead in phase, the slow pendulum 3 shows the smallest amplitude and is slightly behind in phase.
Figure 2.7: Simulation with $n = 3$ pendula, showing states $\theta_i$, $\dot{\theta}_i$ and the accompanying estimated amplitudes $A_i$ and relative time lags $\Delta \tau_i/\tau$ versus time $\tau$. The thin line shows pendulum 1, the medium line pendulum 2 and the thick line pendulum 3.
Figure 2.8: Detail from simulation in Fig. 2.7 with \( n = 3 \) pendula, after some time ended up in an in-phase synchronized steady state, showing estimated amplitudes \( A_i \) and relative time lags \( \Delta \tau_i / \tau \) versus time \( \tau \). The thin line shows pendulum 1, the medium line pendulum 2 and the thick line pendulum 3.
Chapter 3
Experiments

In this section the experimental setup will be treated. This setup is used to test the model derived in Section 2 and to observe the dynamic behavior of multiple coupled pendula.

3.1 Determination of synchronization conditions

3.1.1 Experimental setup

The pendula used in the experiments are Wittner metronomes from the Taktell Super-Mini series 880 (ruby red, model no. 884 051). Their frequency range is from 40 to 208 beats per minute (BPM). The pendulum frequency is set by sliding the bob, a small mass (5.5 gram), along the pendulum. If the bob is removed from the pendulum, the pendulum frequency is approximately 309 BPM. The metronomes tick twice every period, which should be accounted for when converting frequencies from BPM to rad/s.

To compensate for friction to keep the pendulum swinging a mechanical escapement mechanism is present. A spring is wound up manually by the user to store energy. This energy is transferred via wheels, gears, and ratchets, which give the pendulum a small kick each time it swings past a critical angle (approximately $15^\circ$). This can be heard clearly by a loud ticking sound twice each period. A single pendulum swinging stand-alone on a rigid base swings at an amplitude of approximately $45^\circ$ for all pendulum frequency settings and it runs for 10 - 20 minutes depending on the set pendulum frequency. A stand-alone pendulum will die out if the swinging amplitude is smaller than the mentioned critical angle, because the escapement mechanism will not be engaged.

The metronomes are placed on top of a wooden board (98 gram), which is supported by two empty 33 cl beer/soda cans (regular European size, 13 gram each). The cans roll on a flat, level surface. Figure 3.1 shows the setup with $n = 5$ pendula.

It should be noted that it is the actual pendulum frequency $\omega_{pen,i}$ which is set by sliding the bob. This pendulum frequency $\omega_{pen,i}$ is different from the natural frequency $\omega_i$ (from Eq. (2.15)). A single pendulum is only approximately swinging at $\omega_i$ if it is swings at a
very small amplitude and thus becomes approximately a linear system. For larger amplitude (which is the case in the setup) however, the actual pendulum frequency \( \omega_{pen,i} \) decreases. The pendulum frequency \( \omega_{pen,i} \) can be expressed in the natural frequency \( \omega_i \) to first order accuracy as follows:

\[
\omega_{pen,i} = \omega_i \left(1 - \frac{\theta^2}{16}\right),
\]

which is the result of a perturbation analysis from [12], page 94.

To test if the calculation of the mean coupling strength \( \bar{\beta} \) is done correctly a comparison is made with [5]. Using the parameter values calculated in Appendix B the mean coupling strength for the 208 BPM double pendulum system from [5] is calculated as \( \bar{\beta} \approx 0.011 \) (if the wooden board would be 58 g instead, as in [5], which is not the case in these experiments, and \( \omega_1 \approx \omega_2 \)). This value matches the mean coupling strength determined in [5]. This is a strong indication that the calculation of \( \bar{\beta} \) is done correctly.

## 3.1.2 Measurement of synchronization conditions

To estimate conditions for in-phase phase-lock synchronization from the experimental setup, the pendulum frequencies \( \omega_{pen,i} \) are set at equidistant intervals. For an experiment with certain \( \omega_{pen} \) and \( n \), a good starting point for the interval between the different individual pendulum frequencies is to take values close to the bounds for this particular setting as predicted by the simulations.
The frequency differences between the preset marks on the metronome pendulum are too large for most of the experiments. Therefore an iterative manual procedure is used to set the pendula to the right frequencies for each experiment.

Suppose \( n \) is uneven, first the middle pendulum is set to \( \omega_{\text{pen}} \) by timing 20 periods (40 ticks) with a stopwatch. This measurement is done a while after it is started, swinging in a steady state.

Now the pendulum next to it can be set to the slightly different frequency. Both pendula are started (uncoupled) and when they are swinging in a steady state the amount of periods for the faster pendulum to overtake the slower one by one period is counted. This can be done by listening to their ticks, start a stopwatch when they match in-phase and stop it when they match again in-phase.

For instance, it could take 68 periods for the faster one to overtake the slower one, which in his turn has finished 67 periods in the same time. If this ratio 68:67 is not near the desired ratio for the experiment, the faster pendulum has to be re-adjusted by shifting the slider by a small distance (< 1 mm). This procedure can be repeated until the desired frequency difference is set. For the other pendula the same procedure holds. For even \( n \) the procedure is likewise.

After the pendula are set at the right frequencies an experiment can be done. By starting the metronomes a number of times for different initial conditions the occurrence of synchronization for a particular frequency interval, for the particular setting of \( \omega_{\text{pen}} \) and \( n \), for any initial condition can be presumed.

Then by repeating the experiment for the same setting, but with different frequency intervals, bounds for synchronization can be found.

Frequency ratios up to 79:80 are considered feasible to set up properly with the available resources.

The whole experimental procedure is quite cumbersome and can take several hours for large number of pendula \( n \), smaller frequency intervals or lower mean frequency \( \omega_{\text{pen}} \). It is for this reason that in Section 4 the number of experimentally determined synchronization bounds is rather low.

### 3.1.3 Example of an experiment

For \( n = 3 \) and \( \omega_{\text{pen}} = 208 \) BPM, the simulations predict that the transition from synchronization to no synchronization is in between the setting 212, 208, 204 BPM and 213, 208, 203 BPM. So the ideal frequency spacing for starting an experiment for the lower bound would be \( \omega_{\text{pen},1} : \omega_{\text{pen},2} : \omega_{\text{pen},3} = 53 : 52 : 51 \) and for the upper bound approximately \( \omega_{\text{pen},1} : \omega_{\text{pen},2} : \omega_{\text{pen},3} = 43 : 42 : 41 \).

The first experiment is carried out with \( \omega_{\text{pen},1} : \omega_{\text{pen},2} = 68 : 67 \) and \( \omega_{\text{pen},2} : \omega_{\text{pen},3} = 65 : 64 \). Indeed synchronization is observed for a reasonable number of initial conditions. For the second experiment the frequency ratios \( \omega_{\text{pen},1} : \omega_{\text{pen},2} = 46 : 45 \) and \( \omega_{\text{pen},2} : \omega_{\text{pen},3} = 50 : 49 \) are used and no synchronization is observed for a reasonable number of initial conditions.
Although it is impossible to test all initial conditions, the assumption is made that if for a reasonable number of initial conditions each time in-phase synchronization is observed, the setup is in a regime where synchronization does occur. If for a reasonable number of initial conditions each time no in-phase synchronization is observed, the setup is assumed to be in a regime where no synchronization occurs.

The transition from the ‘synchronization’ to the ‘no synchronization’ regime happens somewhere between the two settings found at the edge of the regimes. Thus an estimate for the lower and upper bound are found for this particular setting. In this example the two regimes are found with only two experiments, but it is just as well possible that there are more than two experiments needed to find both regimes. We have to accept that only estimates for the bounds are found, since we cannot do infinitely many experiments.

3.2 Experiments to measure pendulum behavior

3.2.1 Measurement setup

To measure the pendulum motions, a measurement setup is made using very light (0.14 g) accelerometers from Brüel & Kjær. Figure 3.2 shows the setup as it is used. The signals are sampled using a digital analyzer from Brüel & Kjær (input/output front-end 3560C with input/output module 3109), which is controlled using the ‘modal test consultant’ from Brüel & Kjær’s PULSE analyzer software.

![Figure 3.2: Measurement setup for $n = 3$, used to construct phase data by measuring pendulum accelerations.](image)

The accelerometers are fixed to the pendula with accelerometer wax at the big mass, as shown in Fig. 3.3 on the left. Although the accelerometers are very small and the wires are very flexible, they still have a considerable influence on the pendulum behavior. Once wired
up, a metronome that used to swing at 207 BPM, swings at 219 BPM and a metronome that used to swing at 103 BPM, swings at 111 BPM.

For this reason the measurement setup is not suited to investigate synchronization conditions, but it can be used to show general synchronization behavior and other interesting effects (see Section 4.2.1).

To be able to check on correctness of measurement results, one normal sized accelerometer is attached at the base (the wooden board, see Fig. 3.3 on the right) as well, but its data is not used in any plot.

![Figure 3.3: Details of measurement setup showing on the left side the small pendulum accelerometers and on the right side the normal sized base motion accelerometer.](image)

### 3.2.2 Data processing

Because the relative phase differences $\Delta \tau_i/\tau$ between the pendula are an indicator for synchronization, these are the most interesting parameters. Because we are not interested in the correct size of the amplitudes $A_i$, the accelerometers are not calibrated. This explains the difference in scale between the two measured accelerations $\ddot{\theta}_1$ and $\ddot{\theta}_2$. This is not a problem for the extraction of the phases.

The measured acceleration signal is very smooth and after subtracting its mean to correct for any biasing, it can be used directly for $\ddot{\theta}_i$. To reconstruct the velocity $\dot{\theta}_i$, the corrected $\ddot{\theta}_i$ is numerically integrated. This causes some slow low frequent drifting of the signal. To eliminate this the integrated signal is filtered through a 1 Hz - 100 Hz bandpass filter. Finally to make easy phase extraction possible the signal is multiplied with $\omega_{pen}$, which yields $\dot{\theta}_i$. This is based upon the assumption that the signals are nearly sinusoidal.

Another integration would lead to even more drifting of the signal, so $\dot{\theta}_i$ and $\ddot{\theta}_i$ are used to extract phase information and relative time lags (as in Section 2.6) instead of $\theta_i$ and $\dot{\theta}_i$. 

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3.2.3 Measurement example

In Fig. 3.4 the result of a measurement is shown for two pendula beating at around 208 BPM. In both the amplitude and relative time lag plots the beating of the pendula can be observed. After beating for tens of seconds they synchronize to a steady state of in-phase phase-lock.

Figure 3.4: Experiment for $n = 2$ and $\omega_{pen} = 208$ BPM, showing measured accelerations $\ddot{\theta}_i$, reconstructed velocities $\dot{\theta}_i$ and estimates for relative time lags $\Delta \tau_i / \tau$ versus time $t$ ($t$ begins at 8.1 s). After tens of seconds in-phase synchronization occurs.
Chapter 4

Results from experiments and simulations

4.1 Synchronization conditions

There is a strong expectation that a relation between the characteristic coupling strength of the \(n\)-pendula setup and the allowed width of the distribution exists. To investigate this, the critical distribution width is to be found for different settings of mean coupling strengths \(\beta\) and number of pendula \(n\).

To determine which relative frequency differences are allowed between the metronomes for occurrence of in-phase synchronization (phase-lock), numerical simulations are conducted and these are tested with a number of experiments. The pendulum frequencies are spaced at a constant interval for each run at each setting, so they are by definition uniformly distributed.

4.1.1 How the data is treated

To make the amount of data comprehensible the mean pendulum frequency of the particular setting is used:

\[
\bar{\omega}_{\text{pen}} = \frac{1}{n} \sum_{i=1}^{n} \omega_{\text{pen},i},
\]

(4.1)

which allows us to take the standard deviation \(S\) of the different individual pendulum frequencies:

\[
S (\bar{\omega}_{\text{pen}}, n) = \sum_{i=1}^{n} (\omega_{\text{pen},i} - \bar{\omega}_{\text{pen}}),
\]

(4.2)

which can be normalized to \(S^*\) by dividing by \(\bar{\omega}_{\text{pen}}\):

\[
S^* (\bar{\omega}_{\text{pen}}, n) = \frac{S (\bar{\omega}_{\text{pen}}, n)}{\bar{\omega}_{\text{pen}}},
\]

(4.3)
For each setting of $n$ and $\bar{\omega}_{pen}$ a lower bound $S^*_0$ is found beneath which synchronization occurs and an upper bound $S^*_1$, above which synchronization doesn’t occur. Clearly because of a limitation on resources each bound is only found with crude accuracy.

Because the occurrence of phase-lock synchronization is limited to small frequency differences, the differences between the individual coupling strengths $\beta_{ij}$ are also small. Therefore we find the mean of them a viable indicator for general coupling strength of a particular setting:

$$\bar{\beta} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij}, \quad \beta_{ij} > 0. \tag{4.4}$$

Now two essential parameters are defined, $S^*$ and $\bar{\beta}$. These parameters are considered key parameters, which can give valuable insight in the synchronization conditions. In the sequel of this chapter these parameters are used to investigate the relation between the critical distribution width and the mean coupling strength.

### 4.1.2 Range of settings for simulations

Simulations are done for $n = 2, 3, 4$ and $5$ pendula and various mean frequencies $\omega_{pen}$. This leads to a number of simulation settings, which have partly overlap for their average coupling strength $\bar{\beta}$, because the individual coupling strengths $\beta_{ij}$ are dependend from both the number of pendula $n$ and their natural frequencies $\omega_i$. The settings with their resulting mean coupling strengths (calculated using Eq. (2.19)) are shown in Tab. 4.1.

<table>
<thead>
<tr>
<th>$\bar{\omega}_{pen}$ [BPM]</th>
<th>$\bar{\omega}_{pen}$ [rad/s]</th>
<th>$\bar{\beta}$</th>
<th>$\bar{\beta}$</th>
<th>$\bar{\beta}$</th>
<th>$\bar{\beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>236</td>
<td>12.4</td>
<td>0.015</td>
<td>0.011</td>
<td>0.0088</td>
<td>0.0074</td>
</tr>
<tr>
<td>208</td>
<td>10.9</td>
<td>0.0096</td>
<td>0.0073</td>
<td>0.0058</td>
<td>0.0049</td>
</tr>
<tr>
<td>184</td>
<td>9.6</td>
<td>0.0065</td>
<td>0.0049</td>
<td>0.0039</td>
<td>0.0033</td>
</tr>
<tr>
<td>160</td>
<td>8.4</td>
<td>0.0041</td>
<td>0.0031</td>
<td>0.0025</td>
<td>0.0021</td>
</tr>
<tr>
<td>138</td>
<td>7.2</td>
<td>0.0026</td>
<td>0.0019</td>
<td>0.0015</td>
<td>0.0013</td>
</tr>
<tr>
<td>120</td>
<td>6.3</td>
<td>0.0016</td>
<td>0.0012</td>
<td>0.0010</td>
<td>0.00083</td>
</tr>
<tr>
<td>104</td>
<td>5.4</td>
<td>0.0010</td>
<td>0.00075</td>
<td>0.00061</td>
<td>0.00051</td>
</tr>
</tbody>
</table>

Table 4.1: Range of different pendulum frequencies $\bar{\omega}_{pen}$ and number of pendula $n$ and their corresponding mean coupling strengths $\bar{\beta}$, over which numerical simulations are conducted.

### 4.1.3 Example for a particular setting

In an experiment for $n = 3$ and $\bar{\omega}_{pen} = 160$ BPM (which gives $\bar{\beta} = 0.0031$) the following observation is made. If the difference between the successive $\omega_{pen,i}$ is less than 2.0 BPM (so the setting 158, 160, 162 BPM), synchronization does occur. We use Eq. (4.3) and find...
\(S_0(160, 3) = 2.0\) and \(S_0^*(160, 3) = 0.0125\). If the difference between the successive \(\omega_{\text{pen},i}\) is larger than 2.5 BPM (so the setting 157.5, 160, 162.5 BPM), no synchronization occurs. In the same way we calculate \(S_1(160, 3) = 2.5\) and \(S_1^*(160, 3) = 0.0156\).

### 4.1.4 Experimental testing

A number of experiments have been carried out to test the predictions from the numerical simulations. Experiments are carried out for \(n = 2, 3\), and \(\overline{\omega}_{\text{pen}} = 208, 184, 160\) BPM and the data is treated in the same way as the simulation data. Because of the cumbersome experimental procedure (described in Section 3.1.2) the experiments are limited to \(n = 2, 3\).

In the next section the results of these experiments are compared with the simulations.

### 4.1.5 Results on synchronization conditions

The lower and upper bounds \(S_0^*\) and \(S_1^*\) for phase-lock synchronization found in the simulations and the experiments are plotted against the corresponding mean coupling strength \(\overline{\beta}\) in Fig. 4.1. Due to the crudeness of the simulations and experiments, the upper and lower bounds are so far apart.

From the plots it can be observed that the experimental data matches the simulation predictions to a high extent: all experimentally determined bounds overlap the bounds found by simulation. This makes results from the simulations more trustworthy.

Care should be used in interpretation of all plots in Figs. 4.1 and 4.2: the location of the transition points \(S^*\) could be anywhere between each of the bounds, since there is no data available between these bounds. For instance, at first inspection of the bounds from both experiments and simulations in Fig. 4.1b, \(S^*\) seems to grow, reach a maximum and then decay. Indeed for small \(\overline{\beta}\) there is growth, but for both the experiments and simulations holds that the upperbound of the last data point is still larger than the lowerbound of the next to last data point: the conclusion of reaching a maximum and then decay is not justified here.

Each plot in Fig. 4.1 shows an increasing trend. This is in line with the expectation mentioned in Section 4.1: a stronger coupling will probably allow for larger differences between individual oscillator frequencies for synchronization to occur.

An interesting observation is that each plot in Fig. 4.1 shows a steep incline for lower mean coupling strengths and a sudden drop in steepness above a certain mean coupling strength. It seems that there are at least two different regimes, where the coherence between \(S^*\) and \(\overline{\beta}\) is very different. This is in line with the outcome of an analysis for the two pendula setup in section B of [5], of which Fig. 5 shows a similar result. In [11] a simulation study using 800 oscillators (supercritical Hopf bifurcations, also with a uniform frequency distribution), shows the same typical division in coherence (Section 3, Figs. 17 and 18a).
Figure 4.1: Conditions on distribution width $S^*$ for in-phase synchronization versus coupling strength $\beta$ for a) $n = 2$, b) $n = 3$, c) $n = 4$, d) $n = 5$. Beneath dash-dotted lines, connecting lower bound values $S^*_0$, synchronization does occur. Above dashed lines, connecting upper bound values $S^*_1$, no synchronization occurs. Solid lines connect the bounds for each setting. Small markers show numerical results, large markers experimental results.
It can be expected that there is at least one transition point somewhere between each lower and upper bound, where the steady state of in-phase phase-lock synchronization vanishes. A closer look at Fig. 17 reveals that there is actually a transition area, which is very thin for smaller coupling strength, but increases in thickness for larger coupling strength.

Figure 4.2 shows all values from the four plots of Fig. 4.1 in one plot on a double logarithmic scale. The figure shows that all plots have a similar overall increasing tendency.

Another observation, that might not only be derived directly from these plots, but also from Eq. (2.18) is that rising the number of oscillators \( n \), decreases the mean coupling strength \( \beta \) and thus allows for lower frequency distribution width \( S^* \): more oscillators lead to more stringent conditions for in-phase phase-lock synchronization to occur. An additional observation is that the four different plots in Fig. 4.1 also indicate that increasing the number of pendula makes the mean coupling strength, at which the typical jump in steepness occurs, move to the left. This indicates a dependency of the group dynamics on the number of pendula \( n \), which might be interesting for future research.

For a correct interpretation of Figs. 4.1 and 4.2, it should be mentioned that the occurrence of synchronization can depend on initial conditions. Due to the large amount of simulations, all of them are carried out with the same initial conditions and therefore the figures might look slightly different if many different initial conditions were tried for each setting of \( n \) and \( \bar{\omega}_{pen} \).

\section*{4.2 Additional results}

\subsection*{4.2.1 Beating behavior and anti-phase synchronization}

Beating is the slow effect that appears when two or more pendula are trying to synchronize. The pendula exchange energy by slowly changing their amplitudes in opposite directions. The amplitude of the pendulum swinging ahead in phase will grow to slow it down. This increase in amplitude will happen at the cost of a decline in amplitude of the pendulum swinging behind in phase, which makes this pendulum speed up. Clear examples of this behavior are shown in the simulation in Fig. 2.7 and in the experiment in Fig. 3.4.

Both simulations and experiments show that the changes in amplitude during beating are more violent for higher \( \bar{\omega}_{pen} \) (and thus for higher \( \beta \)). In Fig. 4.3 a measurement for \( n = 2 \) and \( \bar{\omega}_{pen} = 309 \) BPM (bobs removed, \( \beta = 0.037 \)) is shown. The system fails to synchronize and keeps on beating.

In experiments for \( n = 3 \) the maximum amplitude of the beating pendula for \( \bar{\omega}_{pen} = 208 \) BPM is so large (around 80°) that they start hitting the bounds for some initial conditions.

However experiments with the setting \( n = 2, 3 \) pendula, \( \bar{\omega}_{pen} = 236 \) BPM (\( \beta = 0.015, 0.011 \) are still carried out. In-phase synchronization occurs only for special initial conditions (chosen already close to the in-phase synchronized state) and small frequency distribution width.
Figure 4.2: Conditions on distribution width $S^*$ for in-phase synchronization versus coupling strength $\beta$, assembly of plots from Fig. 4.1. Solid lines connect lower bound values $S^*_0$, beneath which synchronization does occur and upper bound values $S^*_1$, above which no synchronization occurs. Small markers show numerical results, large markers experimental results. Triangles for $n = 2$, circles for $n = 3$, squares for $n = 4$ and diamonds for $n = 5$. Connecting lines are omitted.
Figure 4.3: Experiment for $n = 2$ and $\omega_{pen} = 309$ BPM ($\beta = 0.037$), showing measured accelerations $\ddot{\theta}_i$ (not at scale) and estimates for relative time lags $\Delta \tau_i/\tau$ versus time $t$ ($t$ begins at 8.1 s). The thin line shows pendulum 1, the thick line pendulum 2. Pendula keep on beating and fail to end up synchronized.
Although no experiments have been carried out to search for anti-phase synchronization conditions, during experiments at higher frequency and coupling strength an interesting observation was made.

For an experiment with \( n = 2 \) pendula and frequency ratio 82:81, for most initial conditions no in-phase synchronization occurs, but long time intervals of anti-phase synchronization exist (20 - 30 periods). These anti-phase semi steady-state situations seem to be almost stable, but after some time they loose their synchrony and start beating for around 20 periods to arrive again at this semi steady state of anti-phase synchronization.

For higher coupling strengths it seems that there is a limit cycle for anti-phase synchronization in the system which gains stability over the limit cycle for in-phase synchronization. The anti-phase limit cycle seems to get a more dominant role in dynamic behavior.

It would be interesting to investigate this behavior for \( n = 2 \) as well as for larger \( n \).

Another effect that has been observed is for experiments with \( n = 3 \) and \( \omega_{\text{pen}} = 309 \) BPM. Figure 4.4 shows how two pendula alternate in beating, while a third one (not shown in the plot) keeps on swinging normally.

### 4.2.2 Frequency-locking

Frequency-locking is the occurrence of two or more subsystems to synchronize their frequencies to a natural ratio of each other, such that they show periodic behavior. This is most likely to happen if their natural frequencies are already nearly at a natural ratio of each other. Experiments have been done where a number of metronomes is set at 208 BPM and synchronize in-phase and one metronome is set at a frequency which is almost a natural ratio of the others. After some settling time, frequency-locking has been observed for the following cases.

Using \( n = 4 \) metronomes, three at 208 BPM, one at 104 BPM, frequency-locking is observed at a ratio of 2:1: Once the system is in a steady state, the three fast pendula make two cycles, while the slow pendulum makes one cycle in the same time.

Using \( n = 5 \) metronomes, four at 208 BPM, one at 139 BPM, frequency-locking is observed at a ratio of 3:2: Once the system is in a steady state, the four fast pendula make three cycles, while the slow pendulum makes two cycles in the same time.

Using \( n = 5 \) metronomes, four at 208 BPM, one at 156 BPM, frequency-locking is observed at a ratio of 4:3: Once the system is in a steady state, the four fast pendula make four cycles, while the slow pendulum makes three cycles in the same time.

This shows that the system of \( n \) pendula can exhibit a much broader range of dynamic behavior than just in-phase (or anti-phase) synchronization.
Figure 4.4: Experiment for $n = 3$ and $\omega_{pen} = 309$ BPM ($\beta = 0.028$), showing measured accelerations $\ddot{\theta}_1$ and $\ddot{\theta}_2$ (not at scale) versus time $t$ ($t$ begins at 8.1 s). Pendula 1 and 2 keep on alternating, while pendulum 3 keeps on swinging normally, it seems like it is not participating in the beating ($\ddot{\theta}_3$ is not shown, since only three accelerometers were used).
4.2.3 Adding mass to lower the coupling strength

A number of experiments is carried out for $n = 2$ and constant $\bar{\omega}_{pen}$, where $\bar{\beta}$ is lowered by adding mass to the base for different distribution widths $S^*$. Now a left bound $\bar{\beta}_0$, to the left of which no in-phase synchronization occurs and a right bound $\bar{\beta}_1$, to the right of which no in-phase synchronization occurs can be measured for various $S^*$. The data from the experiment are in Tab. 4.2. Due to limitations on the setup (no mass can be removed and addition of more than 1 kg will cause damage to the soda cans) the two most extreme values are not determined.

<table>
<thead>
<tr>
<th>$\omega_1:\omega_2$</th>
<th>60:59</th>
<th>44:43</th>
<th>34:33</th>
<th>25:24</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^*$ [-]</td>
<td>0.0120</td>
<td>0.0164</td>
<td>0.0214</td>
<td>0.295</td>
</tr>
<tr>
<td>$\bar{\beta}_0$ [-]</td>
<td>0.0022</td>
<td>0.0036</td>
<td>0.0049</td>
<td>?</td>
</tr>
<tr>
<td>$\bar{\beta}_1$ [-]</td>
<td>?</td>
<td>0.0029</td>
<td>0.0042</td>
<td>0.0103</td>
</tr>
</tbody>
</table>

Table 4.2: Experiment to determine synchronization conditions for $\bar{\omega}_{pen} = 208$ BPM and $n = 2$ pendula. The table shows a range of natural frequency ratios $\omega_1:\omega_2$ and their corresponding distribution width $S^*$. A left bound $\bar{\beta}_0$ to the left of which no synchronization occurs and a right bound $\bar{\beta}_1$ to the right of which synchronization does occur are sought. The coupling strength $\bar{\beta}$ is influenced by addition of mass to the base of the setup.

Figure 4.5: Plot of experiment described in Tab. 4.2, showing distribution width $S^*$ versus coupling strength $\bar{\beta}$. Dash-dotted line connects right bound values, to the right of which synchronization does occur. Dashed line connects left bound values, to the left of which no synchronization occurs. Solid lines connect bounds corresponding to each setting of $S^*$.

A plot of the experiment from Tab. 4.2 is shown in Fig. 4.5. The allowed distribution width $S^*$ for in-phase synchronization decreases, as might be expected, for decreasing $\bar{\beta}$. Again there seems to be a sudden drop in inclination beyond a certain $\bar{\beta}$ as mentioned in Section 4.1.5. This confirms the suspicion of the existence of at least two regimes, where a
different relation exists between \( S^* \) and \( \bar{\beta} \) for the occurrence of synchronization.

Another experimental observation is that the beating pace of the system slows down for weaker coupling strengths. This can be explained by the fact that energy transport from one pendulum to the other cannot take place at the same rate: because the base displacement is smaller (due to the increased mass), forces experienced by the pendula are smaller and thus the beatings slows down noticeably.
Chapter 5

Conclusions and recommendations

5.1 Conclusions

This research, which is part of a traineeship, has the goal of gaining insight in the dynamic behavior of coupled pendula for a setup in which more than two pendula interact. Particular attention is given to conditions for in-phase synchronization to occur, but also a number of additional interesting observations is made.

The experimental setup consists of a light wooden board, which can roll over two soda cans, on top of which up to five piano metronomes can be placed. The setup is used without any measurement device to investigate synchronization conditions. However, there is also a number of experiments done with accelerometers attached to the pendula, to extract detailed information about the phases of the metronomes.

To gain more insight in the synchronization mechanism, a dynamic model is derived, which can be used for an arbitrary number of pendula. This model is implemented in a numerical simulation tool to be able to do predictions on synchronization behavior. Simulations from the tool are compared to experimental results and simulation results from previous research. The tool is also tested with experiments for the prediction of conditions for in-phase synchronization. Based on these tests, model and simulation tool are assumed viable for the prediction of in-phase synchronization.

Several simulations and experiments have been carried out to investigate in-phase phase-lock synchronization conditions for 2, 3, 4 and 5 pendula at different mean frequencies. For each setting (consisting of a certain pendula number and mean frequency) estimates for these conditions have been found by the means of upper and lower bounds. From this research a general proportional relation between mean coupling strength and allowed standardized distribution width is found.

This general relation seems to consist of at least two regimes, where there is a difference in coherence of mean coupling strength and allowed standardized distribution width. The results show that the mean coupling strength, at which this division in coherence is located, decreases for increasing number of pendula.
A number of additional observations has been made experimentally and numerically, which show the interesting rich dynamic behavior of the $n$-pendula setup. The following observations have been made.

For certain settings, large time intervals of approximate anti-phase synchronization occur, while the pendula are not synchronizing in-phase easily. This seems to show that there is a region, where competition between an anti-phase and in-phase limit cycle takes place.

The amplitude of the beating behavior of the pendula, while they try to synchronize, increases for larger mean pendulum frequencies and thus for larger mean coupling strengths. Furthermore, the beating slows down for smaller mean coupling strength, because the pendula are more restricted in their energy transport potential.

Not only pendula which are at approximately the same frequency can synchronize, but also when almost a natural ratio in their frequencies exists. This is observed experimentally for frequency ratios of 2:1, 3:2 and 4:3.

An additional experiment with two pendula has been conducted. In this experiment the coupling strength has been lowered by addition of mass to the base, while the frequency distribution width is kept constant. From this experiment the same relation between mean coupling strength and distribution width appears, showing the same division in coherence: a sudden drop in steepness, beyond a certain mean coupling strength.

Most of these observations give rise to more questions than answers.

No occurence of localization (as in [2], which is mentioned in Section 1) has been observed. Only relatively small differences in steady-state amplitudes are observed, which are still in the same order of magnitude. This can be explained by the fact that localization is a phenomenon that is related to spacious systems, in which wave propagation can take place. Due to the type of setup of this experiment, which involves an all-to-all coupling of the pendula (one can also say that all pendula are connected to one common base), there is no possibility of wave propagation.

5.2 Recommendations

Guidelines for a perturbation analysis on the two coupled pendula setup are given in Appendix A. Work can be done to generalize this analysis for multiple coupled pendula. This could give more insight in synchronization principles and make predictions possible without simulation.

The $n$-pendula simulation tool can be used to predict synchronization for very large $n$, 

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like $n = 100, 1000$. More relationships, for example depending on $n$, could be found. It would also be interesting to investigate more complicated frequency settings, with for instance several natural ratios.

To make measurement of pendulum angles $\theta_i$ easier and at the same time not influence the system, video recordings can be made, which can be analyzed by software. An example in which this has been done successfully is [13].
Bibliography


Appendix A

Analysis using perturbation techniques

Nonlinear systems can often be solved approximately by making use of perturbation methods. These methods are based on the assumption that the variation of amplitudes and phases during one period is small and thus the solution can be found by applying a small nonlinear perturbation to the linearized solution. This makes these methods most suited for weakly nonlinear systems. In this work the method of averaging, one of the many available perturbation methods, is used.

A.1 Method of averaging for two pendula

The equations of motion from Eq. (2.14), rewritten in dimensionless form, can be approximated by a power series including third order nonlinearities as

\[
\ddot{\theta}_1 + (1 + \Delta) \left( \theta_1 - \frac{\theta_1^3}{6} \right) + \mu \left( \frac{\theta_1^2}{\theta_0^2} - 1 \right) \dot{\theta}_1 \\
+ \beta \left[ \ddot{\theta}_1 \left( 1 - \frac{\theta_1^2}{2} \right) - \dot{\theta}_1^2 \theta_1 + \ddot{\theta}_2 \left( 1 - \frac{\theta_2^2}{2} \right) - \dot{\theta}_2^2 \theta_2 \right] = 0, \quad (A.1a)
\]

\[
\ddot{\theta}_2 + (1 - \Delta) \left( \theta_2 - \frac{\theta_2^3}{6} \right) + \mu \left( \frac{\theta_2^2}{\theta_0^2} - 1 \right) \dot{\theta}_2 \\
+ \beta \left[ \ddot{\theta}_1 \left( 1 - \frac{\theta_1^2}{2} \right) - \dot{\theta}_1^2 \theta_1 + \ddot{\theta}_2 \left( 1 - \frac{\theta_2^2}{2} \right) - \dot{\theta}_2^2 \theta_2 \right] = 0, \quad (A.1b)
\]

where \( \beta \) is used for all coupling strengths, assuming \( \beta_{11} \approx \beta_{12} \approx \beta_{21} \approx \beta_{22} \) for small frequency differences and \( \Delta \approx \frac{(\omega_1 - \omega_1^2)}{\omega_{pen}^2} \) is defined as a relative frequency difference.

A coordinate transformation is facilitated by assuming a zeroth order (linear) solution

\[
\theta_1 = A \theta_0 \cos(\tau + \phi); \quad A = A(\tau), \quad \phi = \phi(\tau), \quad (A.2a)
\]

\[
\theta_2 = B \theta_0 \cos(\tau + \xi); \quad B = B(\tau), \quad \xi = \xi(\tau), \quad (A.2b)
\]
for which the amplitude and phase can vary in time. The velocities can be found by taking the time derivative:

\[
\begin{align*}
\dot{\theta}_1 &= \dot{A} \theta_0 \cos \Phi_1 - A \theta_0 (1 + \dot{\phi}) \sin \Phi_1, \\
\dot{\theta}_2 &= \dot{B} \theta_0 \cos \Phi_2 - B \theta_0 (1 + \dot{\xi}) \sin \Phi_2,
\end{align*}
\] (A.3a) (A.3b)

where rapid phases \(\Phi_1 = \tau + \phi\) and \(\Phi_2 = \tau + \xi\) are defined for pendulum 1 and 2.

To complete the transformation the condition is posed that the functional form of the time derivatives of \(\theta_1\) and \(\theta_2\) is similar to the form of the linear solution, where amplitude and phase are constant (the so called Van der Pol transformation), yielding:

\[
\begin{align*}
\dot{\theta}_1 &= A \theta_0 \cos \Phi_1, \\
\dot{\theta}_2 &= B \theta_0 \cos \Phi_2.
\end{align*}
\] (A.4a) (A.4b)

Using Eq. (A.4) and Eq. (A.3)

\[
\begin{align*}
\dot{A} \cos \Phi_1 &= A \dot{\phi} \sin \Phi_1, \\
\dot{B} \cos \Phi_2 &= B \dot{\xi} \sin \Phi_2,
\end{align*}
\] (A.5a) (A.5b)

and

\[
\begin{align*}
\ddot{\theta}_1 &= -\dot{A} \theta_0 \sin \Phi_1 - A \theta_0 (1 + \dot{\phi}) \cos \Phi_1, \\
\ddot{\theta}_2 &= -\dot{B} \theta_0 \sin \Phi_2 - B \theta_0 (1 + \dot{\xi}) \cos \Phi_2.
\end{align*}
\] (A.6a) (A.6b)

If Eq. (A.2), Eq. (A.3) and Eq. (A.6) are substituted in Eq. (A.1), the result is a system of four equations with four unknowns, \(\dot{A}, \dot{B}, \dot{\phi},\) and \(\dot{\xi}\), which can be solved for these unknowns.

A new parameter can be calculated to define the phase difference \(\psi = \phi - \xi\), which leads to

\[
\dot{\psi} = \dot{\phi} - \dot{\xi}
\] (A.7)

which can be expressed in \(A, B\) and \(\psi\) after substitutions. And likewise \(\dot{A}\) and \(\dot{B}\) can be expressed in \(A, B\) and \(\psi\). This is the system stated in Eq. (25) from [5].

In Eqs. (26) and (27) of [5] this system is transformed to another parametrization, which is then used in an analysis to find fixed points and predict synchronization conditions.

### A.2 Method of averaging for multiple pendula

The outlines written in Section C of [5] could be used to do a more thorough analysis on the multiple coupled pendula setup (taking Eq. (2.19) into account) in the same spirit as for the two pendula setup.
Appendix B

Determination of pendulum parameters

By weighing and measuring we can determine the center of mass of pendulum $i$. The pendulum mass without the bob $m_{\text{pen}}$ is 22.8 g, the mass of the bob $m_{\text{bob}}$ is 5.5 g. So if we use the pendula the normal way (with their bobs on) the mass of each pendulum is

$$m_i = m_{\text{pen}} + m_{\text{bob}} = 28.3 \cdot 10^{-3} \text{ kg}. \quad (B.1)$$

The distance between the center of mass of the pendulum without the bob and the pivoting point $r_{\text{pen}}$ is determined to be 16 mm. Thus we can express the distance between the center of mass of the pendulum with its bob and the pivoting point as

$$r_i = \frac{r_{\text{pen}} m_{\text{pen}} - r_{\text{bob}} m_{\text{bob}}}{m_{\text{pen}} + m_{\text{bob}}}. \quad (B.2)$$

Here $r_{\text{bob}}$ is the distance between the bob and the pivoting point. By measuring the BPM scale on the metronome with the use of a flatbed scanner, $r_{\text{bob}}$ can be approximately determined from the BPM rate the metronome is set to. Figure B.1 shows the scale with its preset BPM values. The bob can be placed in between preset values without a problem, so any frequency in the range is possible.

The resulting values $r_i$ for the possible frequencies are shown in Fig. B.2. For easy reference the frequency is also shown in rad/s.

All masses are determined using a ‘Mettler PE 2000’ scale calibrated with ‘Ohaus’ standardized masses.
Figure B.1: Detail of a metronome pendulum and the bob, here separated for illustrative purposes.

Figure B.2: Distance between pendulum center of mass $r$ and pivoting point for different BPM settings